Mathematik

The action of the mapping class group on spaces of metrics of positive scalar curvature

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Abstract

In this thesis we study the action of the θ-mapping class group of a high-dimensional manifold M on the space $\mathcal{R}^+(M)$ of metrics of positive scalar curvature on M via pullback.
We show that this action factors through the bordism group Ω_d^θ of closed θ-manifolds via the mapping torus construction. We then give examples of diffeomorphisms that have null-bordant mapping tori implying that the induced pullback map f^* is homotopic to the identity. Furthermore we show that some old detection results descend from $\mathcal{R}^+(M)$ to the observer moduli space.
Afterwards we construct a family of H-space structures on $\mathcal{R}^+(M)$ for certain M. We show that all of these are isomorphic and use this to derive a criterion for the action map to be trivial up to homotopy. We also give a criterion for the action to be nontrivial leading to a complete classification of the action on simply connected Spin-7-manifolds.

In the last chapter we sketch how one can possibly generalize the recent result about the homotopy groups of $\mathcal{R}^+(M)$ from [BERW17] to a certain class of curvature conditions that imply positive scalar curvature.

Zusammenfassung

In dieser Arbeit untersuchen wir die Wirkung der θ-Abbildungsklassengruppe einer hochdimensionalen Mannigfaltigkeit M auf dem Raum $\mathcal{R}^+(M)$ der Metriken positiver Skalarkrümmung auf M via Rücktransport.
Wir zeigen, dass diese Wirkung über die Abbildungstoruskonstruktion durch die Kobordismusgruppe Ω_d^θ von geschlossenen θ-Mannigfaltigkeiten faktorisiert. Anschließend geben wir Beispiele von Diffeomorphismen, deren Abbildungstori nullbordant sind, was impliziert, dass die induzierte Abbildung f^* homotop zur Identität ist. Des Weiteren zeigen wir, dass einige alte Detektierungsresultate von $\mathcal{R}^+(M)$ auch für den Beobachtermodulraum erhalten bleiben.
Außerdem konstruieren wir eine Familie von H-Raumstrukturen auf $\mathcal{R}^+(M)$ für gewisse M. Wir zeigen, dass all diese isomorph sind, und entwickeln hiermit ein Kriterium dafür, dass die Wirkung bis auf Homotopie trivial ist. Zudem geben wir ein Kriterium für Nicht-Trivialität der Wirkung an, was uns erlaubt die Wirkung auf einfach zusammenhängenden Spin-7-Mannigfaltigkeiten komplett zu klassifizieren.
Im letzten Kapitel skizzieren wir, wie es möglich sein sollte, ein Resultat aus [BERW17] über die Homotopiegruppen von $\mathcal{R}^+(M)$ auf eine gewisse Klasse von Krümmungsbedingungen, die positive Skalarkrümmung implizieren, zu verallgemeinern.

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Introduction

At first sight, the fields of differential geometry and algebraic topology seem substantially different. The former comprises the study of Riemannian metrics, which are fibrewise scalar products on the tangent bundle of a smooth manifold. These metrics are rather sensitive to changes to the manifold. Algebraic topology on the other hand measures coarse properties of a space which are homotopy-, homeomorphism- or diffeomorphism-invariant. However, there is a deep connection between topology and scalar curvature geometry.

The first glimpse of this connection was the discovery of the Lichnerowicz-formula (cf. [Lic63]), relating the difference of the square of the Dirac-operator and the Laplace–Beltrami-operator on spinors to the scalar curvature:

$$\mathcal{D}^2 - \nabla^*\nabla = \frac{\mathbf{scal}}{4}$$

In particular, since $\nabla^*\nabla$ is a positive operator, positivity of the scalar curvature forces the Dirac-operator to be positive. It therefore is invertible making its index vanish if M is closed. In the same year the Atiyah–Singer-index-theorem was proven (cf. [AS63]), providing the possibility to compute the index of the Dirac-operator in terms of topological invariants. To be precise, the index of $\mathcal{D}$ is equal to the $\hat{A}$-genus of M. This means that there is an obstruction to the existence of a positive scalar curvature metric (hereafter: psc-metric) expressed in purely topological terms.

Another connection was discovered independently in the late 1970's by Gromov–Lawson [GL80] and Schoen–Yau [SY79]. They showed that the existence question for a metric of positive scalar curvature is invariant under high-codimension surgeries:

Surgery Theorem ([GL80, Theorem A], [SY79, Corollary 4]). Let M_0, M_1 be smooth manifolds. Let M_0 admit a psc-metric and let M_1 be obtained from M_0 by a sequence of surgeries of codimension at least 3. Then M_1 also admits a psc-metric.

If we assume that $\dim M_1 = d - 1 \geq 5$, we know that M_1 is obtained from M_0 by surgeries in the appropriate dimensions if and only if there exists a manifold W of dimension d such that $\partial W = M_0 \amalg M_1$ and $M_1 \hookrightarrow W$ is 2-connected. The discovery of the surgery theorem dramatically increased the number of manifolds known to admit a psc-metric.

Using tangential structures, it is possible to get rid of the condition on the cobordism: Let $\theta\colon B \to B\mathrm{O}(d)$ be a fibration. A θ-structure on a manifold M is a lift of its Gauss-map along θ. If M_0, M_1 and W admit θ-structures and the map $M_1 \to B$ is 2-connected, one can perform surgery on the interior of W to obtain a cobordism $W'\colon M_0 \rightsquigarrow M_1$ where the inclusion $M_1 \hookrightarrow W'$ is 2-connected, hence the surgery theorem applies. Thus the existence question for psc-metrics can be answered by giving generators of the appropriate cobordism groups that admit metrics of positive scalar curvature.

Examples of tangential structures arise as l-connected covers of $B\mathrm{O}(d)$, which are $B\mathrm{SO}(d)$, $B\mathrm{Spin}(d)$, etc. For example, if M is simply connected and non-spinnable, the map $M_1 \to BSO(d)$ is 2-connected and if M_1 is simply connected and Spin, the map $M_1 \to B\mathrm{Spin}(d)$ is 2-connected. In order to show that a simply connected manifold M_1 admits a psc-metric, it therefore suffices to find a psc-manifold M_0 orientedly cobordant (or Spin-cobordant, respectively) to M_1. For simply connected, non-spinnable manifolds this has been accomplished by Gromov–Lawson who showed that every such manifold of dimension at least 5 admits a psc-metric. Later, Stephan Stolz [Sto92] solved the Spin-case: He was able to determine that the index of the Dirac-operator mentioned above is the *only* obstruction to the existence of a psc-metric on simply connected Spin-manifolds of dimension at least 5. The (stable) Gromov–Lawson–Rosenberg conjecture predicts that a similar statement also holds in the non-simply connected case. There is a counterexample to the unstable conjecture (cf. [Sch98]). The stable conjecture follows from the Baum-Connes conjecture and hence is confirmed for many groups but in general it is still open.

The interest of topologists in positive scalar curvature also goes into another direction: A lot of effort has been put into understanding the homotopy type of the space of all psc-metrics $\mathcal{R}^+(M)$. This all started when Hitchin [Hit74] used index-theory to show that the zeroth and first homotopy group of this space contain nontrivial elements for the standard sphere S^{d-1}. These elements are spotted as follows: For a diffeomorphism f of M and a Riemannian metric g, there exists the pullback metric f^*g on M. If g has positive scalar curvature, so does f^*g and one gets an action of the group of diffeomorphisms $\mathrm{Diff}(M)$ on $\mathcal{R}^+(M)$. Fixing a base point $g_0 \in \mathcal{R}^+(M)$, we get the *orbit map* $\mathrm{Diff}(M) \to \mathcal{R}^+(M)$, $f \mapsto f^*g_0$. Hitchin constructed a homomorphism $\pi_k(\mathcal{R}^+(M)) \to KO^{-k-1-(d-1)}(pt)$, where $d-1$ is the dimension of M, and showed that the composition $\pi_k(\mathrm{Diff}(M)) \to \pi_k(\mathcal{R}^+(M)) \to KO^{-k-d}(pt)$ is nontrivial for $k = 0, 1$ and $M = S^{d-1}$ provided that $k + d \equiv 1, 2(8)$. In other words, nontrivial elements of $\pi_k(\mathcal{R}^+(M))$ are given by nontrivial elements in $\pi_k(\mathrm{Diff}(M))$ that are not in the kernel of the orbit map. Over the years, there have been many other detection results, for example by Gromov–Lawson [GL83], Carr [Car88], Botvinnik–Hanke–Schick–Walsh [BHSW10], Crowley–Schick [CS13], Hanke–Schick–Steimle [HSS14], Crowley–Schick–Steimle [CSS16], Botvinnik–Ebert–Randal-Williams [BERW17] and Ebert–Randal-Williams [ERW17a].

In this thesis we study the homotopy class of the map $\mathrm{Diff}(M) \to \mathrm{h}\mathbf{Aut}(\mathcal{R}^+(M))$ associated to the action mentioned above. Here, $\mathrm{h}\mathbf{Aut}$ denotes the group-like H-space of self-homotopy-equivalences. The main geometric tool we use is a generalization of the surgery theorem due to Chernysh (cf. [Che04b], see also [Wal13]). Let M^{d-1} be a closed manifold and let $\varphi\colon S^{k-1} \times D^{d-k} \hookrightarrow M$ be a surgery datum in M, i. e. an embedding. We denote by M_φ the manifold obtained by performing surgery on M along φ.

Parametrized Surgery Theorem ([Che04b, Theorem 1.1], [Wal13, Main Theorem])**.** *If $d-k \geq 3$, there is a zig-zag of maps $\mathcal{R}^+(M) \overset{\simeq}{\longleftarrow} \dots \longrightarrow \mathcal{R}^+(M_\varphi)$, where the arrow pointing towards $\mathcal{R}^+(M)$ is a weak equivalence.*

Note that $\mathcal{R}^+(M)$ is homotopy equivalent to a CW-complex and hence we can invert all weak equivalences by Whitehead's theorem. So we obtain a well-defined homotopy class of a map $\overline{\mathcal{S}}_\varphi\colon \mathcal{R}^+(M) \longrightarrow \mathcal{R}^+(M_\varphi)$. This is called the *surgery map*.

In order to state our main result, we need to introduce some terminology. For precise definitions see Chapter 1. For a *once-stable*[1] tangential structure $\theta\colon B \to B\mathrm{O}(d)$, we define $\Omega_d^\theta(M_0, M_1)$ to be the set of equivalence classes of triples $[W, f_0, f_1]$, where W is a d-dimensional θ-cobordism with boundary $\partial W = \partial_0 W \amalg \partial_1 W$ and $f_i\colon \partial_i W \overset{\cong}{\longrightarrow} M_i$

[1]A tangential structure is called once stable if it is a homotopy pullback of a fibration $\overline{B} \to B\mathrm{O}(d+1)$ (cf. Section 1.1).

are θ-diffeomorphisms. The equivalence relation is given by the cobordism relation. This gives rise to the groupoid $\hat{\Omega}^{\theta}_{d,2}$: The objects are given by $(d-1)$-dimensional θ-manifolds and the morphism set $\mathbf{mor}_{\hat{\Omega}^{\theta}_{\mathrm{d},2}}(M_0, M_1)$ is defined to be $\Omega^{\theta}_{d}(M_0, M_1)$ if the structure map $M_1 \to B$ is 2-connected and empty otherwise. Furthermore, let hTop denote the homotopy category of spaces. The main theorem of this thesis is the following.

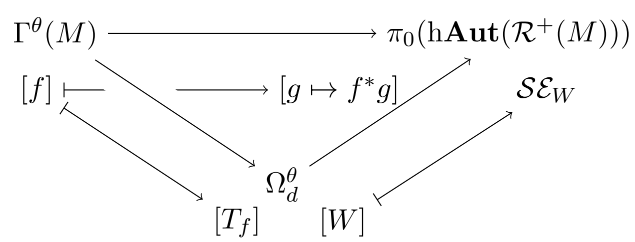
Theorem A (Theorem 3.3.1). *Let $d \geq 7$. Then there is a unique[2] functor $\mathcal{S}\colon \hat{\Omega}^{\theta}_{d,2} \longrightarrow \mathrm{hTop}$ such that*

1. $\mathcal{S}(M) = \mathcal{R}^{+}(M)$,
2. $\mathcal{S}(M \times I, \mathrm{id}, f^{-1}) = [g \mapsto f^{*}g]$,
3. $\mathcal{S}(\boldsymbol{tr}\,\varphi, \mathrm{id}, \mathrm{id}) = \overline{\mathcal{S}}_{\varphi}$ *for* **tr** φ *the trace of a surgery datum* $\varphi\colon S^{k-1} \times D^{d-k} \hookrightarrow M$ *with* $d - k \geq 3$.

Remark. *The definition of the map $\mathcal{S}$ goes back to Walsh (cf. [Wal11] and [Wal14]): He shows that a psc metric g_0 on M_0 can be extended to a metric G on a cobordism $W\colon M_0 \rightsquigarrow M_1$ provided that (W, M_1) is 2-connected. He shows that this construction gives a well defined map $\pi_0(\mathcal{R}^{+}(M_0)) \to \pi_0(\mathcal{R}^{+}(M_1))$ (cf. [Wal14, Theorem 1.3]). The improvement given by Theorem A lies in the following two things: First, instead of a map on π_0 we get a homotopy class of an actual map of spaces and second we show that the map $\mathcal{S}$ is also cobordism-invariant.*

In order to state the most immediate consequence of Theorem A we need some notation. We roughly define the *structured mapping class group* $\Gamma^{\theta}(M)$ to be the components of the groupoid of θ-diffeomorphisms of M (see Section 1.2 for the precise definition) and let $\Omega^{\theta}_{d} \coloneqq \Omega^{\theta}_{d}(\emptyset, \emptyset)$. There is a group homomorphism $\Gamma^{\theta}(M) \to \Omega^{\theta}_{d}$ mapping the homotopy class of a diffeomorphism $[f]$ to its mapping torus denoted by $[T_f]$.

Corollary B (Corollary 4.1.1). *Let $d \geq 7$ and let $\theta\colon B \to B\mathrm{O}(d)$ be the stabilized tangential 2-type of M^{d-1}. Then there is a group homomorphism $\mathcal{SE}\colon \Omega^{\theta}_{d} \to \pi_0\big(\mathrm{h}\mathbf{Aut}(\mathcal{R}^{+}(M))\big)$ such that the following diagram commutes:*



[2]Up to natural isomorphism.

In particular, a θ-diffeomorphism f of M acts trivially on $\mathcal{R}^+(M)$ if its mapping torus is nullbordant.

Let us now give an outline of this thesis.

Chapter 1 provides the necessary background material. We first introduce the notion of tangential structures and give a few important examples in Section 1.1. Afterwards, we define the structured mapping class group and we give two models for it. We also compare it to the mapping class group of orientation preserving diffeomorphisms. Section 1.3 is concerned with cobordism theory. We introduce the cobordism set $\Omega_d^\theta(M_0, M_1)$ of θ-manifolds with fixed boundary θ-diffeomorphic to $M_0 \amalg M_1$. We proceed by showing that this has a free and transitive action from the (structured) cobordism group $\Omega_d^\theta \coloneqq \Omega_d^\theta(\emptyset, \emptyset)$ of closed manifolds. This implies for example that the mapping torus construction gives a homomorphism $\Gamma^\theta(M) \to \Omega_d^\theta$. In the next two Sections (1.4 and 1.5) we discuss handle decompositions of manifolds. In order to compare two of those we first give a recollection on parametrized Morse theory in Section 1.4. We continue by constructing a handle decomposition from a Morse function and we analyze how it changes if one picks a different Morse function. In Section 1.7 we introduce the main object of interest: the space $\mathcal{R}^+(M)$ of psc-metrics on a manifold M. We explain its topology and we state the general version of Chernysh's Parametrized Surgery Theorem along with a few applications.

In the subsequent two chapters we will prove the main theorem. The construction of $\mathcal{S}$ and the proof that $\mathcal{S}$ is well-defined has essentially two steps: First we have to carefully decompose a cobordism into elementary ones, explain how these correspond to surgery data and how two different decompositions are related. This does not involve psc-metrics and we find it best to separate this "cobordism-direction" from the "psc-direction". Having the decomposition at hand we can turn to step two and use the Parametrized Surgery Theorem to define the map $\mathcal{S}$. We now have to study the behavior of $\mathcal{S}$ on psc-metrics and how different decompositions affect the map $\mathcal{S}$.

In **Chapter 2** we start moving in the *cobordism-direction*. After recalling the definition of the unstructured cobordism category $\mathcal{C}ob_d$, we give a slightly different model for $\pi_0(\mathcal{C}ob_d)$ denoted by $\mathcal{B}ord_d$: It has the same objects and a morphism is given by a triple (W, f_0, f_1) consisting of a d-manifold W with decomposable boundary $\partial W = \partial_0 W \amalg \partial_1 W$ and diffeomorphisms $f_i \colon \partial_i W \xrightarrow{\cong} M_i$. Two morphisms (W, f_0, f_1) and (W', f_0', f_1') are identified if there exists a diffeomorphism $F \colon W \xrightarrow{\cong} W'$ that is compatible with the given diffeomorphisms f_i and f_i'. We proceed to give a presentation of $\mathcal{B}ord_d$ in terms of generators and relations. This leads to the notion of the *surgery datum category* $\mathcal{X}_d$: The objects of this category are given by $\mathbf{obj}_{\mathcal{B}\mathrm{ord_d}}$ and morphisms are generated as (possibly empty) strings of composable elementary morphisms of the following form:

1. For a surgery datum $\varphi\colon S^{k-1} \times D^{d-k} \hookrightarrow M$, let $S_\varphi\colon M \to M_\varphi$ be a morphism.
2. For diffeomorphism $f\colon M \xrightarrow{\cong} M'$, let $I_f\colon M \to M'$ be a morphism.

For the relations among these generators see Proposition 1.5.7. We then define wide subcategories $\mathcal{B}ord_d^{a,b}$ and $\mathcal{X}_d^{a,b}$ for $a, b \in \mathbb{N} \cup \{-1\}$ by requiring the following: For an element $(W, f_0, f_1) \in \mathbf{mor}_{\mathcal{B}\mathrm{ord_d^{a,b}}}(M_0, M_1)$ the map $f_0^{-1}\colon M_0 \hookrightarrow W$ is a-connected and $f_1^{-1}\colon M_1 \hookrightarrow W$ is b-connected and for $S_\varphi \in \mathbf{mor}_{\mathcal{X}_\mathrm{d}^{\mathrm{a,b}}}(M_0, M_1)$ we require that for S_φ the surgery datum φ has index $k \in \{a+1, \dots d-b-1\}$. Using the 2-index theorem of Hatcher–Igusa (cf. [Hat75] and [Igu88]) we derive the following result.

Theorem C (Theorem 2.3.3). *For $d \geq 7$ there is an equivalence of categories*

$$\mathcal{P}^{-1,2}\colon \mathcal{X}_d^{-1,2} \to \mathcal{B}ord_d^{-1,2}$$

which is the identity on objects and is given on morphisms by the following:

1. *For $f\colon M_0 \to M_1$, I_f is mapped to $(M_0 \times [0,1], \mathrm{id}, f) \cong (M_1 \times [0,1], f^{-1}, \mathrm{id})$*
2. *For a surgery datum φ in M, S_φ is mapped to $(\boldsymbol{tr}\,(\varphi), \mathrm{id}, \mathrm{id})$.*

Having this result at hand we are able to turn towards the *psc-direction*: In the following **Chapter 3** we define and analyze the *surgery map*

$$\overline{\mathcal{S}}\colon \mathbf{mor}_{\mathcal{B}\mathrm{ord_d^{-1,2}}}(M_0, M_1) \longrightarrow [\mathcal{R}^+(M_0), \mathcal{R}^+(M_1)].$$

It is given by using the presentation from the previous chapter: The morphism I_f is mapped to f_* and for $\varphi\colon S^{k-1} \times D^{d-k} \hookrightarrow M_0$ with $d-k \geq 3$, the morphism S_φ is mapped to the surgery map $\overline{\mathcal{S}}_\varphi\colon \mathcal{R}^+(M) \longrightarrow \mathcal{R}^+(M_\varphi)$. We proceed by showing that this map is well-defined, i. e. that it respects the relations of $\mathcal{X}_d$. We get a well defined homotopy class of a map $\mathcal{S}_W \in [\mathcal{R}^+(M_0), \mathcal{R}^+(M_1)]$ depending only on the diffeomorphism class of W relative to the boundary. Afterwards we show that $\overline{\mathcal{S}}_W$ is invariant under surgeries in the interior of W assuming that these surgeries have the right dimensions and codimensions. Using the fact that the cobordism relation is generated by surgeries this yields the cobordism invariance of the surgery map and thus we have proven Theorem A.

In **Chapter 4** we give several applications of Theorem A and Corollary B. The first one follows immediately from the fact that$\Omega_7^{\mathrm{Spin}} \cong 0 \cong \Omega_7^{\mathrm{SO}}$: Let $\mathrm{Diff}^+(M)$ denote the *group of orientation preserving diffeomorphisms* of M.

Corollary D (Corollary 4.1.3). *Let M^6 be a simply connected manifold. Then the action of* $\mathrm{Diff}^+(M)$ *on* $\mathcal{R}^+(M)$ *is homotopy-trivial, i. e. for every orientation preserving diffeomorphism* f *of* M *the pullback map* f^* *is homotopic to the identity.*

After recalling a few facts about the oriented cobordism ring and the connection to the Spin-cobordism ring we continue by computing cobordism classes of mapping tori and we obtain for example the following implications (for a full list see Section 4.1.2).

Corollary E (Corollary 4.1.21). *Let $d \geq 7$ and let M^{d-1} be a simply connected, closed, oriented manifold. If $d \equiv 0(4)$, let all Pontryagin classes of M vanish. Let $f\colon M \xrightarrow{\cong} M$ be an orientation preserving diffeomorphism. Then $(f^*)^2\colon \mathcal{R}^+(M) \to \mathcal{R}^+(M)$ is homotopic to the identity.*

Being more restrictive on the manifold M, we get a stronger result.

Corollary F (Corollary 4.1.23). *Let $d \geq 7$ and $d \not\equiv 1, 2(8)$. Let M^{d-1} be a simply connected, stably parallelizable manifold. Then the action of* $\mathrm{Diff}^+(M)$ *on* $\mathcal{R}^+(M)$ *is homotopy-trivial.*

This result shows that the detection result of Hitchin [Hit74] is the only possible one of this kind for high-dimensional spheres. An example of an implication for a non-spinnable manifold is the following.

Corollary G (Corollary 4.1.27). *Let X^{2k}, $k \geq 3$ be a stably parallelizable, simply connected, closed manifold and let $H^{2k-i}(X; \mathbb{Z}/2) = 0$ for $i = 3, 5$. Then* $\mathrm{Diff}^+(X \times \mathbb{CP}^2)$ *acts homotopy-trivial on* $\mathcal{R}^+(X \times \mathbb{CP}^2)$.

The next application we present is the canonical follow-up. Having a rigidity result for an action it is natural to ask if one can draw conclusions about the quotient. However, since the action of the diffeomorphism group is not free one has to consider the *observer moduli space* $\mathcal{M}^+_{x_0}(M)$. This is obtained by taking the quotient with respect to the subgroup of those diffeomorphisms f that fix a point $x_0 \in M$ and whose differential

$\mathrm{d}f_{x_0}$ at x_0 is given by the identity. Using the results from the previous sections we show that some of the results from [BERW17] on $\pi_0(\mathcal{R}^+(M))$ descend to $\pi_0(\mathcal{M}^+_{x_0}(M))$:

Theorem H (Theorem 4.2.5). *Let $d \geq 7$ and let M^{d-1} be a 2-connected* Spin*-manifold.*

1. *If $d \equiv 0(4)$ and all Pontryagin classes of M vanish, the space $\mathcal{M}^+_{x_0}(M)$ has infinitely many path components.*
2. *If $d \not\equiv 1, 2(8)$ and M is stably parallelizable, the map $\pi_0(\mathcal{R}^+(M)) \longrightarrow \pi_0(\mathcal{M}^+_{x_0}(M))$ is a bijection.*

Using the work from [GRW16] on the mapping class group of the manifold $W_g^{2n} := (S^n \times S^n)^{\# g}$ we also detect nontrivial elements of $\pi_1(\mathcal{M}^+_{x_0}(W_g^{2n}))$.

Theorem I (Theorem 4.2.6). *For $g \geq 5$, $n \geq 3$ and $n \not\equiv 0(4)$ there is a surjective map $\pi_1(\mathcal{M}^+_{x_0}(W_g^{2n})) \to \Omega^{\langle n \rangle}_{2n+1} \oplus G_n$ where $\Omega^{\langle n \rangle}_{2n+1}$ denotes the $B\mathrm{O}(2n+1)\langle n \rangle$-cobordism group and*

$$G_n \cong \begin{cases} (\mathbb{Z}/2)^2 & \text{if } n \text{ is even} \\ 0 & \text{if } n = 3, 7 \\ \mathbb{Z}/4 & \text{otherwise.} \end{cases}$$

Afterwards we give an application which is a bit more surprising: We use Theorem A to define a homotopy-commutative and homotopy-associative H-space multiplication μ_W on $\mathcal{R}^+(M)$, provided that $W \colon \emptyset \rightsquigarrow M$ is a θ-nullbordism of M. If $W' \colon \emptyset \rightsquigarrow M'$ is another θ-nullbordism of another manifold M', we prove that $\mathcal{S}_{W^{op} \amalg W'}$ is an equivalence of H-spaces. We also show that the path components of invertible elements with respect to μ_W are independent of the choice of W. Using this H-space structure we can also derive the following.

Theorem J (Theorem 4.4.1 and Remark 4.4.2). *Let $d \geq 7$ and let M^{d-1} be a simply connected* Spin*-manifold which is* Spin*-nullbordant. Let f be a* Spin*-diffeomorphism. Then the pullback $f^* \colon \mathcal{R}^+(M) \to \mathcal{R}^+(M)$ is homotopic to the identity if and only if $\mathcal{S}_{S^{d-1} \times [0,1] \amalg T_f}(g_\circ)$ and $g_\circ$ are homotopic in $\mathcal{R}^+(S^{d-1})$, where $g_\circ$ denotes the round metric on S^{d-1}.*

Using an argument in the style of Carr (cf. [Car88]) we also deduce a non-triviality criterion.

Proposition K (Proposition 4.4.3). *Let M be a $(d-1)$-dimensional, simply connected* Spin*-manifold and let W^d be manifold with $\hat{A}(W) \neq 0$. Then $\mathcal{SE}_W(g) \not\sim g$ for every psc-metric g on M. In particular, $\mathcal{SE}_W$ is not homotopic to the identity.*

Using this result we are able to fully classify the action of $\mathrm{Diff}^{\mathrm{Spin}}(M)$ on $\mathcal{R}^+(M)$ for M a simply connected Spin-manifold of dimension 7:

Corollary L (Corollary 4.4.4). *Let M be a 7-dimensional, simply connected Spin-manifold and let $f\colon M \xrightarrow{\cong} M$ be a Spin-diffeomorphism. Then the following are equivalent:*

1. *$\hat{A}(T_f) = 0$.*
2. *T_f is Spin-nullbordant.*
3. *f^* is homotopic to the identity.*
4. *$f^*g \sim g$ for every $g \in \mathcal{R}^+(M)$.*
5. *There exists a metric $g \in \mathcal{R}^+(M)$ such that $f^*g \sim g$.*

The final **Chapter 5** of this thesis is somewhat disconnected from the rest. Recently, Kordaß [Kor18] generalized Chernysh's Parametrized Surgery Theorem to a more general class of curvature conditions. These are called *deformable, codimension c surgery stable curvature conditions*, where $c \geq 3$. We will abbreviate this by dCcSS. We apply Kordaß' result to derive an analogue of our main result for dCcSS which encode *the mixed-torpedo condition*. Most of the proofs go through without change but the dimension restrictions change. Let C be a dCcSS that encodes the mixed torpedo condition and let $\mathcal{R}_C(M)$ denote the space of metrics satisfying C. We get the following result.

Corollary M (Corollary 5.2.10). *Let $d \geq 2c + 1$ and let M^{d-1} be a $(c-2)$-connected $B\mathrm{O}(d)\langle c-1\rangle$ manifold. If $d \equiv 0(4)$ let all Pontryagin classes of M vanish. Let $f\colon M \xrightarrow{\cong} M$ be an orientation preserving diffeomorphism. Then $(f^*)^n\colon \mathcal{R}_C(M) \to \mathcal{R}_C(M)$ is homotopic to the identity for some $n \in \mathbb{N}$.*

Afterwards we indicate how to extend the detection result of Botvinnik–Ebert–Randal-Williams [BERW17] to dCcSS. We first prove a special case of the existence of stable metrics.

Lemma N (Lemma 5.3.1). *Let $d \geq 2c$ and let $V^{d-1}\colon S^{d-2} \rightsquigarrow S^{d-2}$ be a $(c-2)$-connected, $BO(d)\langle c-1\rangle$-cobordism. Also, assume that V is $BO(d)\langle c-1\rangle$-cobordant to $S^{d-2} \times [0,1]$ relative to the boundary. Then there exists a metric $g \in \mathcal{R}_C(V)_{g_\circ,g_\circ}$ with the following property: If $W : S^{d-2} \rightsquigarrow S^{d-2}$ is cobordism and $h \in \mathcal{R}(S^{d-2})$ is a boundary condition such* *that* $h + dt^2 \in \mathcal{R}_C(S^{d-2} \times [0,1])$ *then the two gluing maps*

$$\mu(\_, g)\colon \mathcal{R}_C(W)_{h,g_\circ} \longrightarrow \mathcal{R}_C(W \cup V)_{h,g_\circ}$$
$$\mu(g, \_)\colon \mathcal{R}_C(W)_{g_\circ,h} \longrightarrow \mathcal{R}_C(V \cup W)_{g_\circ,h}$$

are homotopy equivalences.

We use this to show that for a certain class of manifolds M with boundary S^{d-1} the action of $\mathrm{Diff}_\partial(M)$ factors through an abelian group:

Theorem O (Theorem 5.3.2)**.** *Let* $d \geq 2c$ *and let* M^{d-1} *be a* $(c-2)$*-connected,* $B\mathrm{O}(d)\langle c-1\rangle$*-manifold with boundary* $\partial M = S^{d-2}$*. Also, assume that* M *is* $B\mathrm{O}(d)\langle c-1\rangle$*-cobordant to* D^{d-1} *relative to the boundary. Then the image of the action homomorphism* $\pi_0(\mathrm{Diff}_\partial(M)) \to \pi_0(\mathrm{h}\mathbf{Aut}(\mathcal{R}_C(M)_{g_\circ^{d-2}}))$ *is an abelian group.*

Using this it should be possible to generalize the results from [BERW17] to dCcSS with $c = 3, 4$ that imply positive scalar curvature. This would show that the nontrivial elements in $\pi_k(\mathcal{R}^+(M))$ are in the image of the inclusion map $\mathcal{R}_C(M) \hookrightarrow \mathcal{R}^+(M)$.

Contents

1

Preliminaries

1.1 Tangential structures and Moore-Postnikov towers

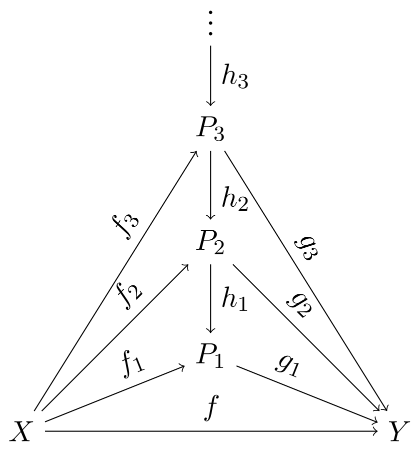
In this section we recall the notion of tangential structures and give the examples most important to us. For $d \geq 0$ let $B\mathrm{O}(d)$ be the classifying space for the d-dimensional orthogonal group and let $U_d := E\mathrm{O}(d) \underset{\mathrm{O}(d)}{\times} \mathbb{R}^d$ be the universal vector bundle over $B\mathrm{O}(d)$. Let $\theta\colon B \to B\mathrm{O}(d)$ be a fibration. We call θ a *tangential structure*.

Definition 1.1.1. A *θ-structure* on a real $\mathrm{rank(d)}$-vector bundle $V \to X$ is a bundle map $\hat{l}\colon V \to \theta^* U_d$. A *$\theta$-structure on a manifold W^d* is a θ-structure on TW and a *θ-manifold* is a pair $(W, \hat{l})$ of a manifold W and a θ-structure $\hat{l}$ on it. For $0 \leq k < d$ a *stabilized θ-structure* on M^k is a θ-structure on $TM \oplus \mathbb{R}^{d-k}$.

An important source of tangential structures are covers of $B\mathrm{O}(d)$. For example we have $B\mathrm{SO}(d) \to B\mathrm{O}(d)$ or $B\mathrm{Spin}(d) \to B\mathrm{O}(d)$ or more generally $B\mathrm{O}(d)\langle k\rangle \to B\mathrm{O}(d)$, where $B\mathrm{O}(d)\langle k\rangle$ denotes the k-connected cover of $B\mathrm{O}(d)$. Other sources of tangential structures are Moore-Postnikov towers.

Definition 1.1.2 ([Hat02, p. 414])**.** Let X, Y be connected spaces and let $f\colon X \to Y$ be a map. A *Moore-Postnikov tower* for f is a collection of tupels $(P_n, f_n, g_n, h_n)_{n\in\mathbb{N}}$, where

P_n are spaces, $f_n \colon X \to P_n$ is an n-connected map, $g_n \colon P_n \to Y$ is an n-coconnected fibration and $h_n \colon P_{n+1} \to P_n$ is a fibration such that the following diagram commutes:



We call (P_n, f_n, g_n, h_n) the *n-th stage of a Moore-Postnikov tower*.

Theorem 1.1.3 ([Hat02, Theorem 4.71]). *Every map $f \colon X \to Y$ between path-connected spaces has a Moore-Postnikov-tower, which is unique up to homotopy equivalence.*

Definition 1.1.4. Let M^{d-1} be a connected manifold, let $l \colon M \to B\mathrm{O}(d)$ be the classifying map of the stabilized tangent bundle and let $\hat{l} \colon TM \oplus \underline{\mathbb{R}} \to U_d$ be a bundle map covering l. The n-th stage of the Moore-Postnikov tower for the map l is called the *stabilized tangential n-type of M*. We write $B_n(M) := P_n$ in this case.

We call θ *once-stable* (see [GRW14, Definition 5.4]) if there exists a map $\overline{\theta} \colon \overline{B} \to B\mathrm{O}(d+1)$ which fits into a (homotopy) pullback diagram as follows:

$$\begin{array}{ccc} B & \longrightarrow & \overline{B} \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\overline{\theta}} \\ B\mathrm{O}(d) & \longrightarrow & B\mathrm{O}(d+1) \end{array}$$

We call θ *n-stable* for $n \geq 1$ if θ is once-stable and $\overline{\theta}$ is $(n-1)$-stable (with the convention that 0-stable is the empty condition). We call θ *stable* if it is n-stable for every $n \geq 1$.

Proposition 1.1.5 ([GRW14, Lemma 5.9]). *If $d - 1 \geq 3$, the tangential structure given by the (stabilized) tangential 2-type of a manifold is stable.*

Example 1.1.6. 1. The tangential 2-type of a connected spin manifold M of dimension at least 3 is $B\mathrm{Spin}(d) \times B\pi_1(M)$.

2. The tangential 2-type of a simply connected, non-spinnable manifold M of dimension at least 3 is $B\mathrm{SO}(d)$.

Proof. The first part is obvious as $B\mathrm{Spin}(d)$ is 2-connected for $d \geq 3$. For the second part we note that the map $l\colon M \to BSO(d) \times B\pi_1(M)$ is an isomorphism on π_1 as $B\mathrm{SO}(d)$ is simply connected. Furthermore, we have the isomorphism $\pi_2 B\mathrm{SO}(d) \to \mathbb{Z}/2\mathbb{Z}$ given by $[\beta] \mapsto \langle w_2(\theta^* U_d), \beta_*[S^2]\rangle$ for $\theta\colon B\mathrm{SO}(d) \to B\mathrm{O}(d)$. The map $l_*\colon \pi_2(M) \to \pi_2(B\mathrm{SO}(d)) \to \mathbb{Z}/2\mathbb{Z}$ sends a class $[\alpha]$ to

$$\langle w_2(\theta^* U_d), l_*\alpha_*[S^2]\rangle = \langle l^* w_2(\theta^* U_d), \alpha_*[S^2]\rangle = \langle w_2(\underbrace{l^*\theta^* U_d}_{\cong TM}), \alpha_*[S^2]\rangle$$

Since M is non-spinnable, it's second Stiefel-Whitney class is nonzero. Furthermore, as M is simply connected, the Hurewicz homomorphism $\pi_2(M) \to H_2(M)$ is surjective and there exists an α such that the above expression is nonzero and $M \to B\mathrm{SO}(d)$ is surjective on π_2. □

1.2 Mapping class groups

In this section we will give the definitions and present two models for the structured mapping class group of a manifold.

Definition 1.2.1. For a smooth manifold M^{d-1} we denote by $\mathrm{Diff}(M)$ the *topological group of diffeomorphisms of* M with the C^∞-topology. If M is oriented we denote *the subgroup of orientation preserving diffeomorphisms of* M by $\mathrm{Diff}^+(M)$. The *(unoriented) mapping class group* $\Gamma(M)$ is defined to be $\pi_0(\mathrm{Diff}(M))$ and the *oriented mapping class group* $\Gamma^+(M)$ is defined as $\pi_0(\mathrm{Diff}^+(M))$.

Definition 1.2.2. Let M^{d-1} be a smooth oriented manifold. We define

$$B\mathrm{Diff}^\theta(M) := E\mathrm{Diff}(M) \underset{\mathrm{Diff}(M)}{\times} \mathrm{Bun}(TM \oplus \underline{\mathbb{R}}, \theta^* U_d),$$

where we use the model $E\mathrm{Diff}(M) := \{j\colon M \hookrightarrow \mathbb{R}^{\infty-1}\}$ which is the (contractible) space of embeddings and $\mathrm{Bun}(\_,\_)$ denotes the space of bundle maps. More concretely,

$$B\mathrm{Diff}^\theta(M) = \{(N, \hat{l})\colon N \subset \mathbb{R}^{\infty-1},\ N \cong M \text{ and } \hat{l} \in \mathrm{Bun}(TN \oplus \underline{\mathbb{R}}, \theta^* U_d)\}.$$

Given an embedding $j\colon M \hookrightarrow \mathbb{R}^{\infty-1}$ and a (stabilized) θ-structure $\hat{l}$ on M, we get a base-point $(j(M),\hat{l}) \in B\mathrm{Diff}^\theta(M)$. Furthermore, we define

$$E^\theta_{\mathrm{Diff}}(M) \coloneqq E\mathrm{Diff}(M) \times \mathrm{Bun}(TM \oplus \underline{\mathbb{R}}, \theta^* U_d).$$

We also define the *universal M-bundle with θ-structure $U_{M,\theta}$* by

$$U_{M,\theta} \coloneqq \Big(E\mathrm{Diff}(M) \times \mathrm{Bun}(TM \oplus \underline{\mathbb{R}}, \theta^* U_d)\Big) \underset{\mathrm{Diff}(M)}{\times} M \longrightarrow B\mathrm{Diff}^\theta(M).$$

Remark 1.2.3. For $\theta_{BSO}\colon B\mathrm{SO}(d) \to B\mathrm{O}(d)$ we abbreviate $B\mathrm{Diff}^{\theta_{BSO}}(M)$ by $B\mathrm{Diff}^+(M)$. Note that with our definition $E\mathrm{Diff}^+(M)$ is not contractible but homotopy equivalent to $\mathrm{Bun}(TM \oplus \underline{\mathbb{R}}, \theta^*_{\mathrm{BSO}} U_d)$ which has two contractible components (cf. Lemma 1.2.6 and Lemma 1.2.7)

Definition 1.2.4 (Structured Mapping Class Group)**.** Let M be a smooth submanifold of $\mathbb{R}^{\infty-1}$ and let $\hat{l}$ be a stabilized θ-structure on M. The *θ-structured mapping class group* $\Gamma^\theta(M,\hat{l})$ is defined by

$$\Gamma^\theta(M,\hat{l}) := \pi_1(B\mathrm{Diff}^\theta(M), (M,\hat{l})).$$

For $\gamma\colon S^1 \to B\mathrm{Diff}^\theta(M)$ we define the *structured mapping torus* $M_\gamma := \gamma^* U_{M,\theta}$.

Remark 1.2.5. The mapping torus M_γ has a fiber wise θ-structure. Since the tangent bundle of the circle is trivial, this gives a θ-structure on M_γ.

Before further analyzing the structured mapping class group let us have a closer look at the space of bundle maps $\mathrm{Bun}(V, \theta^* U_d)$ for a $\mathrm{rank(d)}$ vector bundle $\pi_V\colon V \to X$ over a finite CW-complex X. Let $\hat{\tau}\colon V \to U_d$ be a fixed bundle map and let $\tau\colon V \to B\mathrm{O}(d)$ be the underlying map of spaces. We get an isomorphism of bundles

$$\begin{aligned} \alpha_\tau\colon V &\to \tau^* U_d \coloneqq \{(p,u) \in X \times U_d \colon \tau(p) = \pi_{U_d}(u)\} \\ w &\mapsto (\pi_V(w), \hat{\tau}(w)). \end{aligned}$$

Let $\overline{\tau}\colon \tau^* U_d \to U_d$ and $\overline{\theta}\colon \theta^* U_d \to U_d$ denote the induced maps. Then $\overline{\tau} \circ \alpha_\tau = \hat{\tau}$. Now we define $\mathrm{lifts}_{\tau,\theta} \coloneqq \{l\colon X \to B\colon \theta \circ l = \tau\}$ to be the space of lifts of τ along θ and $\mathrm{Bun}_{\hat{\tau},\overline{\theta}} \coloneqq \{\hat{l} \in \mathrm{Bun}(V, \theta^* U_d)\colon \overline{\theta} \circ \hat{l} = \hat{\tau}\}$ to be the space of lifts of bundle maps. Let z_V denote the zero section of V. We get a map $v\colon \mathrm{Bun}(V, \theta^* U_d) \to \mathrm{Map}(X,B)$ defined by $v(\hat{l}) = \pi_{\theta^* U_d} \circ \hat{l} \circ z_V$. This induces a map $v_{\hat{\tau},\overline{\theta}}\colon \mathrm{Bun}_{\hat{\tau},\overline{\theta}} \longrightarrow \mathrm{lifts}_{\tau,\theta}$.

Lemma 1.2.6. *The map* $v_{\hat{\tau},\overline{\theta}} \colon \mathrm{Bun}_{\hat{\tau},\overline{\theta}} \longrightarrow \mathrm{lifts}_{\tau,\theta}$ *is a homeomorphism.*

Proof. We prove this lemma by constructing an inverse map. Let $l \in \mathrm{lifts}_{\tau,\theta}$. Then we define

$$b(l)\colon V \xrightarrow{\alpha_\tau} \underbrace{\tau^* U_d}_{=l^*\theta^* U_d} \xrightarrow{\bar{l}} \theta^* U_d.$$

and we claim that $l \mapsto b(l)$ is an inverse to the map v. Note that $\bar{l} \colon l^*\theta^* U_d \to \theta^* U_d$ is given by $\bar{l}(p,u) = (l(p), u)$. Then

$$\begin{aligned} v_{\hat{\tau},\overline{\theta}}(b(l))(p) &= \pi_{\theta^* U_d} \circ \bar{l} \circ \alpha_\tau \circ z_V(p) = \pi_{\theta^* U_d}\Big(\bar{l}(p, \hat{\tau}(z_V(p)))\Big) \\ &= \pi_{\theta^* U_d}\Big(l(p), \hat{\tau}(z_V(p))\Big) = l(p) \end{aligned}$$

and so $v_{\hat{\tau},\overline{\theta}} \circ b = id$.

Let $\hat{l} \in \mathrm{Bun}_{\hat{\tau},\overline{\theta}}$. Then for $w \in V$ we have $\hat{l}(w) = (b_{\hat{l}}, u_{\hat{l}})$ for some $(b_{\hat{l}}, u_{\hat{l}}) \in \theta^* U_d$. Note that $b_{\hat{l}} = \pi_{\theta^* U_d} \circ \hat{l} \circ z_V(\pi_V(w))$ and $\overline{\theta}(b_{\hat{l}}, u_{\hat{l}}) = u_{\hat{l}}$. Since $\hat{l}$ is a lift of $\hat{\tau}$ we see that $u_{\hat{l}} = \overline{\theta}(\hat{l}(w)) = \hat{\tau}(w)$. Now we compute

$$\begin{aligned} b(v_{\hat{\tau},\overline{\theta}}(\hat{l}))(w) &= \overline{\pi_{\theta^* U_d} \circ \hat{l} \circ z_V} \circ \alpha_\tau(w) \\ &= \overline{\pi_{\theta^* U_d} \circ \hat{l} \circ z_V}(\pi_V(w), \hat{\tau}(w)) \\ &= \Big(\pi_{\theta^* U_d} \circ \hat{l} \circ z_V(\pi_V(w)), \hat{\tau}(w)\Big) = (b_{\hat{l}}, u_{\hat{l}}) = \hat{l}(w). \end{aligned}$$

□

Lemma 1.2.7. *The map* $\mathrm{Bun}(V, \theta^* U_d) \longrightarrow \mathrm{Bun}(V, U_d)$ *is a Serre fibration.*

Proof. We need to consider the following lifting problem:

$$\begin{array}{ccc} \{0\} \times D & \xrightarrow{\quad A \quad} & \mathrm{Bun}(V, \theta^* U_d) \\ \Big\downarrow & \overset{\tilde{A}}{\nearrow} & \Big\downarrow \\ [0,1] \times D & \xrightarrow{\quad a \quad} & \mathrm{Bun}(V, U_d) \end{array}$$

Using the map v from above we get a diagram

$$\begin{array}{ccc} \{0\} \times D & \xrightarrow{\quad v \circ A \quad} & \mathrm{Map}(X, B) \\ \Big\downarrow & \overset{\tilde{A}'}{\nearrow} & \Big\downarrow \\ [0,1] \times D & \xrightarrow{\quad v \circ a \quad} & \mathrm{Map}(X, BO(d)). \end{array}$$

By Lemma B.1 we know that $\mathrm{Map}(X, B) \to \mathrm{Map}(X, B\mathrm{O}(d))$ is a Serre-fibration and the dashed arrow A' in this diagram exists. We then define

$$A(t,p)(w) := \Big(A'(t,p)(\pi_V(w)), a(t,p)(w)\Big). \qquad \square$$

Since $\mathrm{Bun}(TM \oplus \mathbb{R}, U_d)$ is contractible by the classification of bundles, Lemma 1.2.6 and Lemma 1.2.7 imply that the space $\mathrm{Bun}(TM \oplus \underline{\mathbb{R}}, \theta^*_{BSO} U_d)$ has two contractible components provided that M is orientable.

Now let us continue investigating $\Gamma^\theta(M, \hat{l})$. There is a forgetful map $B\mathrm{Diff}^\theta(M) \to B\mathrm{Diff}(M)$ that induces a map $\Gamma^\theta(M, \hat{l}) \to \Gamma(M) = \pi_0(\mathrm{Diff}(M))$. So, every element $\gamma \in \Gamma^\theta(M, \hat{l})$ has an associated isotopy class of an actual diffeomorphism $f\colon M \to M$. The underlying (unstructured) manifold of the mapping torus M_γ is given by the usual mapping torus T_f of f.

Proposition 1.2.8. *Let $\theta\colon B\mathrm{O}(d)\langle k\rangle \to B\mathrm{O}(d)$ and let M be a $(k-1)$-connected θ-manifold of dimension $d-1 \geq k+1 \geq 3$. Then the forgetful map $B\mathrm{Diff}^\theta(M) \to B\mathrm{Diff}^+(M)$ induces a surjection*

$$\Gamma^\theta(M, \hat{l}) \twoheadrightarrow \Gamma^+(M)$$

Proof. The forgetful map $B\mathrm{Diff}^\theta(M) \to B\mathrm{Diff}^+(M)$ fits into the following diagram of fibrations.

$$\begin{array}{ccccccc}
 & & \mathrm{Diff}(M) & = & \mathrm{Diff}(M) & & \\
 & & \downarrow & & \downarrow & & \\
\mathrm{Bun}(TM \oplus \underline{\mathbb{R}}, \theta^* U_d) & \xleftarrow{\sim} & E\mathrm{Diff}^\theta(M) & \longrightarrow & E\mathrm{Diff}^+(M) & \xrightarrow{\sim} & \mathrm{Bun}(TM \oplus \underline{\mathbb{R}}, \theta^*_{\mathrm{BSO}} U_d) \\
 & & \downarrow & & \downarrow & & \\
 & & B\mathrm{Diff}^\theta(M) & \longrightarrow & B\mathrm{Diff}^+(M) & &
\end{array}$$

The map $\pi_1(E\mathrm{Diff}^\theta(M)) \longrightarrow \pi_1(E\mathrm{Diff}^+(M))$ is surjective, because components of $\mathrm{Bun}(TM, \theta^*_{BSO} U_d)$ are contractible. Next, let us consider the map $B\mathrm{O}(d)\langle m\rangle \to B\mathrm{O}(d)\langle m-1\rangle$ for $2 \leq m \leq k$. Its homotopy fiber F is given by

$$\Omega K(\pi_m B\mathrm{O}(d), m) \simeq K(\pi_m B\mathrm{O}(d), m-1) = K(\pi_m B\mathrm{O}, m-1).$$

By our assumption on M there exists a lift of $M \to B\mathrm{O}(d)\langle m-1\rangle$ to $B\mathrm{O}(d)\langle m\rangle$. The obstructions to the uniqueness up to homotopy of such a lift lies in the groups (cf.

[Hat02, pp. 418])

$$H^n(M, \pi_n(F)) \cong \begin{cases} H^{m-1}(M, \pi_m B\mathrm{O}) & \text{if } n = m-1 \\ 0 & \text{else.} \end{cases}$$

Since M is $(k-1)$-connected and $m \geq 2$ all these groups vanish and so the lift to $B\mathrm{O}(d)\langle m\rangle$ is unique. From Lemma 1.2.6 and Lemma 1.2.7 we deduce that for every $[\hat{l}_{BSO}] \in \pi_0(\mathrm{Bun}(TM \oplus \underline{\mathbb{R}}, \theta^*_{BSO}U_d))$ there exists a unique lift $[\hat{l}] \in \pi_0(\mathrm{Bun}(TM \oplus \underline{\mathbb{R}}, \theta^* U_d))$. Therefore the map $\pi_0(E\mathrm{Diff}^\theta(M)) \longrightarrow \pi_0(E\mathrm{Diff}^+(M))$ is injective and the Proposition follows from the first half of the 5-Lemma. □

Let us have a closer look at the case of $B = B\mathrm{Spin}(d)$. Let us recall the more traditional description of Spin-structures (cf. [Ebe06, Chapter 3]): *A* Spin*-structure σ on a manifold* M is a pair (P, α) consisting of a $\mathrm{Spin}(d)$-principal bundle P and an isomorphism $\alpha\colon P \times_{\mathrm{Spin}(d)} \mathbb{R}^d \xrightarrow{\cong} TM \oplus \underline{\mathbb{R}}$. An *isomorphism of* Spin*-structures* $\sigma_0 = (P_0, \alpha_0)$ and $\sigma_1 = (P_1, \alpha_1)$ is an isomorphism $\beta\colon P_0 \xrightarrow{\cong} P_1$ of $\mathrm{Spin}(d)$-principal bundles over id_M such that $\alpha_1 \circ (\beta \times_{\mathrm{Spin}(d)} \mathrm{id}_{\mathbb{R}^d}) = \alpha_0$. If $f\colon M \to M$ is an orientation preserving diffeomorphism and $\sigma = (P, \alpha)$ is a Spin structure on M, we define $f^*\sigma := (f^*P, (df)^{-1} \circ f^*\alpha)$.

The first naive idea of a definition of $\mathrm{Diff}^{\mathrm{Spin}}(M, \sigma)$ is the following:

$$\mathrm{Diff}^{\mathrm{Spin}}_{naive}(M, \sigma) := \{f \in \mathrm{Diff}^+(M)\colon \sigma \cong f^*\sigma\} \subset \mathrm{Diff}^+(M).$$

However this cannot work as illustrated by the following example: Consider $M = pt$. This has precisely one spin structure and one self-diffeomorphism, so $\mathrm{Diff}^{\mathrm{Spin}}_{naive}(pt)$ is a point, and hence so is its classifying space $B\mathrm{Diff}^{\mathrm{Spin}}_{naive}(pt)$. Homotopy classes of maps $S^1 \to B\mathrm{Diff}^{\mathrm{Spin}}_{naive}(pt)$ should classify spin structures on the circle (as point bundles over S^1). However, there are 2 non-isomorphic Spin-structures on the circle. So, this is not the correct automorphism space of a Spin-manifold. Also note, that $B\mathrm{Diff}^{\mathrm{Spin}}(M)$ is possibly not connected, so $\mathrm{Diff}^{\mathrm{Spin}}(M)$ cannot be a group.

The correct definition of $\mathrm{Diff}^{\mathrm{Spin}}(M, \sigma)$ is the following (cf. [Ebe06, Definition 3.3.3]): Let σ_0, σ_1 be two Spin-structures of M. A Spin-diffeomorphsim $(M, \sigma_0) \xrightarrow{\cong} (M, \sigma_1)$ is a pair $(f, \hat{f})$ consisting of an orientation preserving diffeomorphism $f\colon M \xrightarrow{\cong} M$ and an isomorphism $\hat{f}$ of Spin-structures σ_0 and $f^*\sigma_1$. We denote by $\mathrm{Diff}^{\mathrm{Spin}}((M, \sigma_0), (M, \sigma_1))$ the set of Spin diffeomorphisms $(M, \sigma_0) \xrightarrow{\cong} (M, \sigma_1)$. This gives rise to the groupoid

$\mathrm{Diff}^{\mathrm{Spin}}(M)$ which has Spin structures on M as objects and morphisms sets are given by $\mathrm{Diff}^{\mathrm{Spin}}((M,\sigma_0),(M,\sigma_1))$. For a Spin-structure σ on M, we define

$$\mathrm{Diff}^{\mathrm{Spin}}(M,\sigma) \coloneqq \mathrm{Diff}^{\mathrm{Spin}}((M,\sigma)(M,\sigma)).$$

We can now give an easier proof for and actually a strengthening of Proposition 1.2.8.

Proposition 1.2.9. *Let M be a simply connected* Spin *manifold. Then the forgetful homomorphism* $\mathrm{Diff}^{\mathrm{Spin}}(M,\sigma) \to \mathrm{Diff}^{+}(M)$ *is surjective and its kernel has two elements.*

Proof. Since M is simply connected, the Spin-structure σ of an oriented manifold is unique up to isomorphism. So for every orientation preserving diffeomorphism $f\colon M \xrightarrow{\cong} M$, there is an isomorphism $\sigma \xrightarrow{\cong} f^*\sigma$, hence the map is surjective. The rest follows from [Ebe06, Lemma 3.3.6]. □

It would be conceptually satisfying to also give a description of a groupoid $\mathrm{Diff}^\theta(M)$ that has $B\mathrm{Diff}^\theta(M)$ as a classifying space and whose elements are related to diffeomorphisms. The correct one is the one which has bundle maps $\hat{l}\colon TM \oplus \underline{\mathbb{R}} \to \theta^* U_d$ as objects. A morphism is a pair (f, L) consisting of a diffeomorphism f and a path L of bundle maps connecting $\hat{l}_0$ and $\hat{l}_1 \circ df$. However, concatenation of paths is only associative up to homotopy, so one would need to consider this as an ∞-groupoid. In order to give a model for the mapping class group one does not need the language of ∞-categories, though.

Definition 1.2.10. For a θ-structure $\hat{l}$ on M^{d-1} we define

$$B^\theta(M,\hat{l}) \coloneqq \left\{ (f,L)\colon \begin{array}{l} f\colon M \xrightarrow{\cong} M \text{ is a diffeomorphism} \\ L \text{ is a homotopy of bundle maps } \hat{l}\circ df \rightsquigarrow \hat{l} \end{array} \right\} \Big/ \sim$$

where the equivalence relation is given by homotopies of f and L.

We want to compare $\Gamma^\theta(M,\hat{l})$ and $B^\theta(M,\hat{l})$. First note that $B\mathrm{Diff}^\theta(M)$ classifies M-bundles with θ-structure. So, elements in $\Gamma^\theta(M,\hat{l})$ are precisely given by fiber bundles over $[0,1]$ with θ-structures on the vertical tangent bundle that restricts to $(M,\hat{l})$ over the points $\{0,1\} \in [0,1]$.

For a representative (f,L) of an element in $B^\theta(M,\hat{l})$ we get such a bundle $E \to [0,1]$ by gluing $[0,\frac{1}{2}] \times M$ to $[\frac{1}{2},1] \times M$ along f and the θ-structure is given by $\hat{l}$ on $[0,\frac{1}{2}] \times M$ and by the rescaled path L on $[\frac{1}{2},1] \times M$. Replacing f and L by homotopic maps results in an isomorphic bundle E' and so we get a map $P\colon B^\theta(M,\hat{l}) \longrightarrow \Gamma^\theta(M,\hat{l})$ defined by $P(f,L) = E$.

There is also a map in the other direction. Let $E \to [0,1]$ be an M-bundle with θ-structure $\hat{l}_E$ on the vertical tangent bundle of E. Since the interval is contractible, there exists a diffeomorphism $\psi\colon E \xrightarrow{\cong} [0,1]\times M$ that fibers over the interval and restricts to the identity on $\{0\}\times M$. We get a diffeomorphism $\psi_1 := \psi(1, \_)$ of M. Furthermore, $L_t := (\hat{l}_E \circ d\psi_t^{-1})|_{\{t\}\times M}$ defines a path of bundle maps from $\hat{l}$ to $\hat{l}\circ d\psi_1$. We get a map $Q\colon \Gamma^\theta(M,\hat{l}) \xrightarrow{\cong} B^\theta(M,\hat{l})$ by $Q(E) = (\psi_1^{-1}, L)$. This construction is invariant with respect to isomorphism of the bundle E.

Proposition 1.2.11. *The above constructions give mutually inverse group isomorphisms between $\Gamma^\theta(M,\hat{l})$ and $B^\theta(M,\hat{l})$.*

Proof. Let $[f,L] \in B^\theta(M,\hat{l})$. Then there is a diffeomorphism $\psi\colon P(f,L) \xrightarrow{\cong} [0,1]\times M$ given by the identity on $[0,\frac{1}{2}]\times M$ and by $f^{-1}\times \mathrm{id}$ on $[\frac{1}{2},1]\times M$. Then $\psi_1^{-1} = f$ and the bundle homotopy constructed above is homotopic to L by contracting $[0,\frac{1}{2}]\times M$ and stretching $[\frac{1}{2},1]\times M$.

Now, let $[E] \in \Gamma^\theta(M,\hat{l})$. Let $\psi\colon E \xrightarrow{\cong} [0,1]\times M$ and L_t be as above. Then $Q(\psi_1^{-1}, L)$ is isomorphic to E by contracting $[\frac{1}{2},1]\times M$ and stretching $[0,\frac{1}{2}]\times M$ composed with ψ^{-1}. □

Example 1.2.12. Since we usually will be interested in the case where θ is the (stabilized) tangential 2-type of a high-dimensional manifold M, let us as have a closer look at the case of $B = B\mathrm{Spin}(d)\times BG$. The map $\theta\colon B\mathrm{Spin}(d)\times BG \to B\mathrm{O}(d)$ factors through the 3-connected cover $\theta_{\mathrm{Spin}}\colon B\mathrm{Spin}(d) \to B\mathrm{O}(d)$ and we get

$$\mathrm{Bun}(TM\oplus\underline{\mathbb{R}}, \theta^* U_d) = \mathrm{Bun}(TM\oplus\underline{\mathbb{R}}, \theta^*_{\mathrm{Spin}} U_d)\times \mathrm{Map}(M, BG).$$

So, a θ-structure $\hat{l}$ on M is given by a Spin structure σ on M and a map $M\to BG$. Let $\psi := [f,L] \in B^\theta(M,\hat{l})$. Then f is an orientation preserving diffeomorphism of M and L is the homotopy class of a path connecting the bundle maps $\hat{l}_{\mathrm{Spin}}, \hat{l}_{\mathrm{Spin}}\circ df\colon TM\oplus\underline{\mathbb{R}} \to \theta^*_{\mathrm{Spin}}U_d$ together with the homotopy class of a path connecting the maps α and $\alpha\circ f\colon M\to BG$.

If $G = \pi_1(M,x)$ for some base-point $x\in M$, this means that the induced map $f_*\colon \pi_1(M,x) \to \pi_1(M,f(x))$ is given by conjugation by a path $\gamma\colon [0,1]\to M$ with

$\gamma(0) = x$ and $\gamma(1) = f(x)$. We say that f *acts on the fundamental group by an inner automorphism* in this case.

1.3 Cobordism groups

In this section we define the θ-structured cobordism set of manifolds with fixed boundary. We compare it to the θ-structured cobordism group of closed manifolds. We assume that all cobordisms are ε-collared for some ε.

Definition 1.3.1. Let $\theta\colon B \to B\mathrm{O}(d)$ be a once-stable tangential structure and let M_0^{d-1}, M_1^{d-1} be compact manifolds with (stabilized) θ-structures l_0, l_1. We define the *bordism set of manifolds with θ-structure and fixed boundary* by

$$\Omega_d^\theta\big((M_0, \hat{l}_0), (M_1, \hat{l}_1)\big) := \big\{(W, \psi_0, \psi_1, \hat{\ell})\big\}/\sim .$$

Here, W is a d-manifold with boundary $\partial W = \partial_0 W \coprod \partial_1 W$, $\hat{\ell} \in \mathrm{Bun}(TW, \theta^* U_d)$ is a bundle map and $\psi_i = (f_i, L_i)$, $i = 0, 1$ are θ-diffeomorphisms, i. e. $f_i\colon \partial_i W \to M_i$ are diffeomorphisms and L_i are homotopies of bundle maps $(-1)^i \hat{l}_i \circ \mathrm{df} \sim \hat{\ell}|_{\partial_i W}$, where $-\hat{l}_1$ denotes the bundle map $\hat{l}_1 \circ (\mathrm{id} \oplus (-1))\colon TM \oplus \underline{\mathbb{R}} \to TM \oplus \underline{\mathbb{R}} \to \theta^* U_d$. We call M_0 the *incoming boundary* and M_1 the *outgoing boundary* (see Figure 1.1).

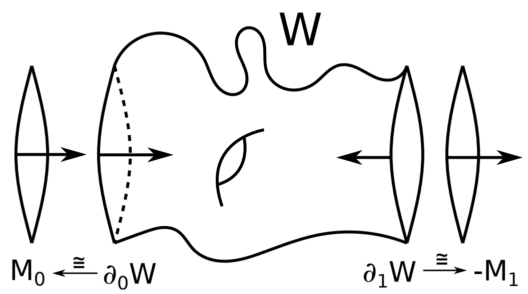


FIGURE 1.1: A representative of $[W] \in \Omega_d^\theta\big((M_0, \hat{l}_0), (M_1, \hat{l}_1)\big)$.
The arrows indicate the direction of the θ-structures.

The equivalence relation is given by the cobordism relation: We say that $(W, \psi_0, \psi_1, \ell)$ and $(W', \psi_0', \psi_1', \ell')$ are cobordant if there exists a $d+1$-dimensional θ-manifold (X, ℓ_X) with corners such that there exists a partition of $\partial X = \bigcup_{i=0,3} \partial_i X$ together with

θ-diffeomorphisms

$$M_0 \times I \cong \partial_0 X \qquad M_1 \times I \cong \partial_3 X$$
$$W \cong \partial_2 X \qquad W' \cong \partial_1 X$$

such that θ-structures and diffeomorphisms fit together (see Figure 1.2).

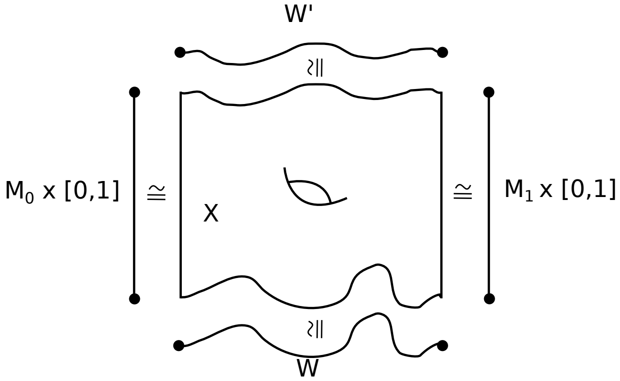


FIGURE 1.2: The cobordism relation in $\Omega_d^\theta\big((M_0, \hat{l}_0), (M_1, \hat{l}_1)\big)$.

Remark 1.3.2.

1. Since θ is a fibration we can arrange the θ-structure on W so that the θ-structures $(-1)^i \hat{l}_i \circ \mathrm{df_i}$ and $\hat{\ell}|_{\partial_i W}$ actually agree.

2. Let θ be once-stable. $\Omega_d^\theta((M, \hat{l}), (M, \hat{l}))$ becomes a monoid via concatenation of cobordisms and identifying them along the boundary diffeomorphisms, i. e.

$$(W', \psi_0', \psi_1', \ell') \cdot (W, \psi_0, \psi_1, \ell) := (W \cup_{f_0' \circ f_1^{-1}} W', \psi_0, \psi_1', \ell \cup_{\hat{f}_0' \circ \hat{f}_1^{-1}} \ell').$$

We will later see that this monoid is actually is a commutative group (cf. Corollary 1.3.7). More generally, this gives rise to a category $\hat{\Omega}_d^\theta$ with objects $(M, \hat{l})$ and morphism set $\Omega_d^\theta\big((M_0, \hat{l}_0), (M_1, \hat{l}_1)\big)$.

3. Note that one has a map $\Omega_d^\theta((M, \hat{l}), (M, \hat{l})) \to \Omega_d^\theta(\emptyset, \emptyset) =: \Omega_d^\theta$ given by identifying the *equal* boundaries of a cobordism $W \colon M \rightsquigarrow M$. This map gives an isomorphism of groups (cf. Corollary 1.3.7 and the remark below it).

The following proposition will be very useful later on.

Proposition 1.3.3. *Let θ be a once-stable tangential structure and let $W^d \colon M_0 \rightsquigarrow M_1$ be θ-cobordism. Then there exists a θ-structure on $W^{\mathrm{op}} \colon M_1 \rightsquigarrow M_0$ such that $W \cup W^{\mathrm{op}} \sim$*

$M_0 \times [0,1]$ rel $M_0 \times \{0,1\}$. In particular, if $W \colon \emptyset \rightsquigarrow M$ is a θ-nullbordism, the double $dW := W \cup W^{\mathrm{op}}$ is nullbordant and $W^{\mathrm{op}} \amalg W$ is cobordant to the cylinder on M.

Proof. Consider the manifold with corners $W \times I$. We introduce new corners as in Figure 1.3.

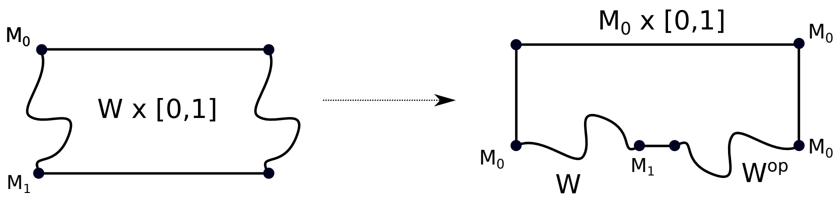


FIGURE 1.3: Introducing corners to obtain the desired cobordism

Let us now construct the θ-structures[1]. Let $\overline{\theta} \colon \overline{B} \to BO(d+1)$ be as in the definition of once-stable. We get a $\overline{\theta}$-structure $\bar{l}_W \colon TW \oplus \underline{\mathbb{R}} \to \theta^* U_{d+1}$. Since $W \hookrightarrow W \times [0,1]$ is a homotopy equivalence there is a unique extension up to homotopy

$$\begin{array}{ccc} TW \oplus \underline{\mathbb{R}} & \xrightarrow{\quad \bar{l}_W \quad} & \overline{\theta} \\ \downarrow & \nearrow & \\ T(W \times [0,1]) & & \end{array}$$

where the vertical map sends $v \in \underline{\mathbb{R}}_{>0}$ to the inwards pointing vector. This gives a $\overline{\theta}$-structure on $W \times I$ and by restriction a θ-structure on W^{op} as θ is once-stable. □

We obtain the following.

Corollary 1.3.4. *Let θ be once-stable. Then the the category $\hat{\Omega}^{\theta}_d$ is a groupoid. In particular $\Omega^{\theta}_d\big((M,\hat{l}),(M,\hat{l})\big)$ is a group.*

Now we prove another useful tool.

Proposition 1.3.5. *Let θ be once-stable. Then the action of Ω^{θ}_d on $\Omega^{\theta}_d\big((M_0,l_0),(M_1,l_1)\big)$ given by disjoint union is free and transitive.*

[1]This is adapted from [GRW14, Proof of Proposition 2.16].

Proof. Since disjoint union is associative up to cobordism and disjoint union with the emptyset is the identity and this really defines a group action.

If $\Omega_d^\theta\big((M_0, l_0), (M_1, l_1)\big) = \emptyset$ the statement is trivial. So let $(L, \psi_0^L, \psi_1^L, \ell_L) \colon (M_0, \hat{l}_0) \rightsquigarrow (M_1, \hat{l}_1)$ be a cobordism. Let $\Phi_L \colon \Omega_d^\theta \longrightarrow \Omega_d^\theta\big((M_0, l_0), (M_1, l_1)\big)$ be given by $\Phi_L(V) = V \amalg L$. Also let

$$\tilde{\Phi}_L \colon \Omega_d^\theta\big((M_0, l_0), (M_1, l_1)\big) \longrightarrow \Omega_d^\theta$$

be defined given by gluing in the cobordism $(L^{\mathrm{op}}, \psi_1^L, \psi_0^L, \ell_L^{\mathrm{op}})$ along the boundary as follows: We concatenate with L^{op} and then identify the equal boundaries:

$$\Omega_d^\theta\big((M_0, l_0), (M_1, l_1)\big) \overset{\cup L^{\mathrm{op}}}{\longrightarrow} \Omega_d^\theta\big((M_0, l_0), (M_0, l_0)\big) \longrightarrow \Omega_d^\theta$$

We will prove the Proposition by showing that Φ and $\tilde{\Phi}$ are mutually inverse bijections. The easy part is

$$\tilde{\Phi}(\Phi([V])) = \tilde{\Phi}([V \amalg L]) = [V \amalg (L \cup L^{\mathrm{op}})] = [V]$$

by Proposition 1.3.3. It remains to show that $(W \cup L^{\mathrm{op}}) \amalg L$ is cobordant to W. First we note that (W, ψ_0, ψ_1) is diffeomorphic to $(M_0 \times I \cup_{\psi_0} W \cup_{\psi_1^{-1}} M_1 \times I, \mathrm{id}, \mathrm{id})$ and so it suffices to consider the case that all boundary identifications are given by the identity. We now decompose $\tilde{\Phi}_L(W) \amalg L = (W \cup L^{\mathrm{op}}) \amalg L$ as follows:

$$V_0 := M_0 \times [0, \varepsilon] \cup M_1 \times [1-\varepsilon, 1] \qquad V_1 := L$$
$$V_2 := L^{\mathrm{op}} \qquad V_3 := W$$

Note that

$$\partial V_0 = (M_0 \times \{0\}) \amalg \underbrace{(M_0 \times \{\varepsilon\}) \amalg (M_1 \times \{1-\varepsilon\})}_{=:\partial_+ V_0} \amalg (M_1 \times \{1\})$$
$$\partial V_1 = M_0 \amalg M_1 = \partial V_2 = \partial V_3$$

By identifying $\partial_+ V_0$ and ∂V_2 with ∂V_1 and ∂V_3 in different ways we obtain

$$V_0 \cup V_1 = L \qquad V_2 \cup V_3 = L^{\mathrm{op}} \cup W$$
$$V_0 \cup V_3 = W \qquad V_2 \cup V_3 = L^{\mathrm{op}} \cup L = dL$$

We will now give a cobordism $X \colon (V_0 \cup V_1) \amalg (V_2 \cup V_3) \rightsquigarrow (V_0 \cup V_3) \amalg (V_2 \cup V_1)$.

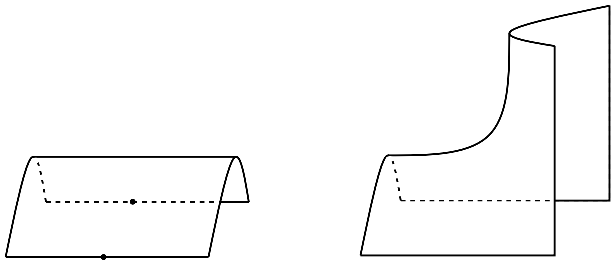


FIGURE 1.4: Introducing corners at the boundary of $V_i \times [0,1]$

We construct this by taking $V_i \times I$ for every $i = 0,1,2,3$, introducing corners at the boundary (and at $\partial_+ V$) respectively) as shown in Figure 1.4 and gluing the obtained manifolds together along parts of the boundary as shown in Figure 1.5. The θ-structures are given by $\hat{l}_i \oplus \mathrm{id}_{\underline{\mathbb{R}}}$ (the arrows in Figure 1.5 indicate the incoming and outgoing boundary of X). □

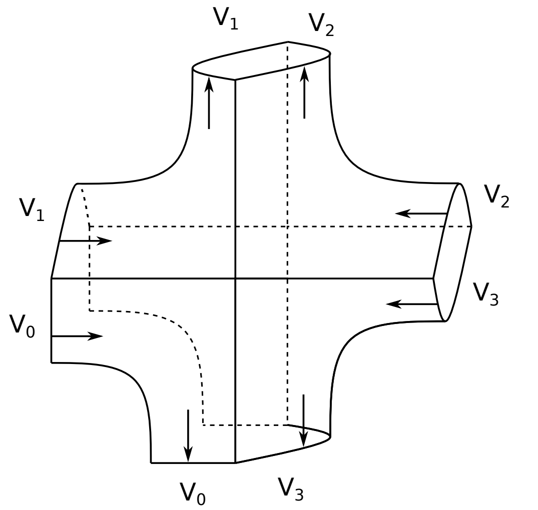


FIGURE 1.5: The cobordism $X\colon (V_0 \cup V_1) \amalg (V_2 \cup V_3) \rightsquigarrow (V_0 \cup V_3) \amalg (V_2 \cup V_1)$

Remark 1.3.6. Proposition 1.3.5 can also be proven using structured cobordism categories. The presented proof however is much more direct.

As a corollaries we get:

Corollary 1.3.7. *Let (M,l) be a $(d-1)$-dimensional θ-manifold. Then the map*

$$\Phi\colon \Omega_d^\theta \xrightarrow{\cong} \Omega_d^\theta\big((M,\hat{l}),(M,\hat{l})\big)$$

given by $(V, \hat{\ell}) \mapsto (M \times [0,1] \amalg V, \mathrm{id}, \mathrm{id}, (\hat{l} \oplus \mathrm{id}_{\mathbb{R}}) \amalg \hat{\ell})$ *is an isomorphism of groups. In particular,* $\Omega_d^\theta\big((M, \hat{l}), (M, \hat{l})\big)$ *is an abelian group.*

Proof. It is a group homomorphism because

$$\begin{aligned}\Phi(V \amalg W) &= M \times [0,1] \ \amalg \ V \amalg W \\ &= (M \times [1,2] \ \amalg \ V) \cup (M \times [0,1] \ \amalg \ W) = \Phi(V) \cup \Phi(W).\end{aligned}$$

The rest follows from Proposition 1.3.5. □

Remark 1.3.8. The inverse map is given by mapping (W, ψ_0, ψ_1) to the manifold obtained by gluing $\partial_1 W$ to $\partial_0 W$ along the diffeomorphism $\psi_0^{-1} \circ \psi_1$.

Corollary 1.3.9. *The map* $\Gamma^\theta(M, \hat{l}) \to \Omega_d^\theta$ *given by* $[\gamma] \mapsto [M_\gamma]$ *is a homomorphism.*

Proof. Consider $\gamma\colon [0,1] \to B\mathrm{Diff}^\theta(M)$ as a path from $(M, \hat{l})$ to itself. We define the *mapping cylinder map* by $A\colon \Gamma^\theta(M, \hat{l}) \to \Omega_d^\theta(M, M), \gamma \mapsto (\gamma^* U_{M,\theta}, \mathrm{id}, \mathrm{id})$. Since the bundle classified by $\gamma_0 * \gamma_1$ is the same as the union of the bundles classified by γ_i, this satisfies

$$\begin{aligned}A(\gamma_0 * \gamma_1) &= ((\gamma_0 * \gamma_1)^* U_{M,\theta}, \mathrm{id}, \mathrm{id}) \\ &= (\gamma_0^* U_{M,\theta} \cup \gamma_1^* U_{M,\theta}, \mathrm{id}, \mathrm{id}) \\ &= (\gamma_0^* U_{M,\theta}, \mathrm{id}, \mathrm{id}) \cup (\gamma_1^* U_{M,\theta}, \mathrm{id}, \mathrm{id}) = A(\gamma_0) \cup A(\gamma_1).\end{aligned}$$

Since the isomorphism $\Omega_d^\theta(M, M) \to \Omega_d^\theta$ is given by gluing the boundary, we have $M_\gamma = \tilde{\Phi}(\gamma^* U_{M,\theta})$ and hence

$$M_{\gamma_0 * \gamma_1} = \tilde{\Phi}(A(\gamma_0 * \gamma_1)) = \tilde{\Phi}(A(\gamma_0)) \amalg \tilde{\Phi}(A(\gamma_1) = M_{\gamma_0} \amalg M_{\gamma_1}.$$ □

Remark 1.3.10. Using the model $B^\theta(M, \hat{l})$ for the mapping class group, we see that the map $A \circ P\colon B^\theta(M, \hat{l}) \to \Omega_d^\theta(M, M)$ is given by $(f, L) \mapsto (M \times [0,1], \mathrm{id}, (f, L)^{-1})$ for P the map from Proposition 1.2.11. Note that since $\Omega_d^\theta(M, M)$ is commutative, $(f, L) \mapsto (M \times [0,1], \mathrm{id}, (f, L))$ is a homomorphism as well.

1.4 Morse theory

In this section we recall the basic notions from Morse theory and parametrized Morse theory.

Definition 1.4.1. Let $W^d\colon M_0 \rightsquigarrow M_1$ be cobordism with collars. A *Morse function* $f\colon W \to [0,1]$ is a smooth function such that

1. It is collared, i. e. $f^{-1}([0,\varepsilon)) = [0,\varepsilon) \times M_0$, $f^{-1}((1-\varepsilon,1]) = (1-\varepsilon,1] \times M_1$ for some $\varepsilon > 0$ and $f(t,p) = t$ for $t \in [0,\varepsilon) \cup (1-\varepsilon,1]$.
2. For every critical point $p \in W$, $\mathrm{d}^2 f_p$ has rank d. These points are called *nondegenerate critical points*.

The following is the well-known Morse-lemma, see for example [Mat02, Theorem 2.16].

Lemma 1.4.2. *Let $p \in W$ be a nondegenerate critical point of a Morse function f. Then there exists a chart $k\colon U \subset \mathbb{R}^d \to W$ centered at p (i. e. $0 \in U$ and $k(0) = p$) and $\lambda \in \{0,\dots,d\}$ satisfying*

$$f \circ k(x_1,\dots,x_d) = f(p) + \sum_{i=1}^{\lambda} -x_i^2 + \sum_{i=\lambda+1}^{d} x_i^2$$

We call k a Morse chart *and λ the* index of f at p.

Later on, we want to investigate the handle decompositions associated to two different Morse functions. To this end, we need to understand how to compare Morse functions. The space of all Morse functions is not connected in general which means in order to connect two Morse functions by a path one has to allow further singularities.

Lemma 1.4.3 ([Igu88, Lemma 1.5, p.298])**.** *Let $W\colon M_0 \rightsquigarrow M_1$ be a cobordism and let $f\colon W \to [0,1]$ be a smooth function such that $f^{-1}([0,\varepsilon)) = [0,\varepsilon) \times M_0$ and $f^{-1}((1-\varepsilon,1]) = (1-\varepsilon,1] \times M_1$ for some $\varepsilon > 0$. Assume that there also exists a critical point $p \in W$ of f such that $\mathrm{rank}(\mathrm{d}^2\mathrm{f}_\mathrm{p}) = d-1$. Then there exists a chart $k\colon U \subset \mathbb{R}^d \to W$ centered at p such that $f \circ k(x_1,\dots,x_d) = f(p) + r(x_1) + q(x_2,\dots,x_d)$, where $q = \sum_{i=2}^{\lambda} -x_i^2 + \sum_{i=\lambda+1}^{d} x_i^2$ and $r\colon \mathbb{R} \to \mathbb{R}$ is a smooth function.*

Definition 1.4.4. We call a singularity $p \in W$ as in Lemma 1.4.3 an *A_l-singularity* if $r^{(i)}(0) = 0$ for $i \leq l$ and $r^{(l+1)}(0) \neq 0$ and we define $\lambda - 1$ to be the index of f at p. We call an A_2-singularity a *birth-death-singularity*. If all critical points of f are A_1 or A_2 singularities, we call f a *generalized Morse function*.

Lemma 1.4.5 ([Igu88, Theorem 1.7, p. 301])**.** *Let p be an A_l singularity of $f\colon W \to [0,1]$. Then there exists a chart $k\colon U \subset \mathbb{R}^d \to W$ centered at p such that*

$$f \circ k(x_1, \dots, x_d) = f(p) + x_1^{l+1} + \sum_{i=2}^{\lambda} -x_i^2 + \sum_{i=\lambda+1}^{d} x_i^2$$

By abuse of notation, we again call k a Morse chart centered at *p.*

Definition 1.4.6. Let $f_t\colon W \to [0,1]$ be a path of generalized Morse functions and let p be a birth-death-singularity of f_0. Let $k\colon \mathbb{R}^d \to W$ be a Morse chart centered at p. We call p *generically unfolded by* f_t if $\frac{\partial^2}{\partial t \partial x_1} f_t(0) \neq 0$.

Definition 1.4.7. Let $h\colon I \times W \to [0,1]$ be a collared path of generalized Morse functions, i. e. $h_t(x) = h_{t'}(x)$ for all x in a neighbourhood of ∂W. We call this path *generic*, if there exist disjoint finite subsets $T_0, T_1 \subset I$ such that

1. $h(t, \_)$ is a Morse function for $t \notin T_0$.
2. For $t \in T_0$, the function $h(t, \_)$ has precisely one birth-death-singularity which is generically unfolded.
3. $h(t, \_)$ has distinct critical values for $t \notin T_1$.
4. If $t \in T_1$ then $h(t, \_)$ has precisely 2 critical points p_0, p_1 with agreeing critical values. These shall satisfy $\frac{\partial}{\partial t} h(t, p_0) \neq \frac{\partial}{\partial t} h(t, p_1)$

We call $T = T_0 \cup T_1$ the *set of singular times* of h_t.

Remark 1.4.8. For a path of generalized Morse functions there exists the so-called Cerf-Kirby-graphic which depicts the movement of the critical values along the time axis (See Figure 1.6)

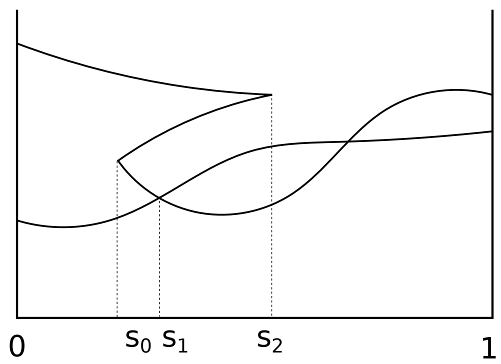


FIGURE 1.6: A Cerf-Kirby-graphic

Lemma 1.4.9 ([Igu88, Lemma 2.5, p. 308])**.** *If p is a birth-death singularity of f_0 generically unfolded by f_t, there exists an $\varepsilon > 0$, a chart k centered at p and path of generalized Morse functions h_t isotopic to f_t by an isotopy supported in a small neighbourhood of p such that*

$$h_t \circ k(x_1, \dots, x_d) = f(p) + x_1^3 \pm tx_1 + \sum_{i=2}^{\lambda} -x_i^2 + \sum_{i=\lambda+1}^{d} x_i^2$$

for $t \in (-\varepsilon, \varepsilon)$.

The following Lemma is due to Cerf [Cer70]. Since the original source is in french, we decided to give a proof in the appendix for convenience of author and reader.

Lemma A.2. *Let $h_0, h_1 \colon W \to [0,1]$ be Morse functions with distinct critical values. Then there exists a generic path of generalized Morse functions connecting them.*

1.5 Handle decompositions of cobordisms

In this section we discuss the relation between Morse functions on W and handle decompositions of W. First, we give a model for attaching a handle. The one given in [Per17, Construction 8.1] is convenient.

Construction 1.5.1 (Standard trace)**.** Let $\varepsilon \in (0, \frac{1}{4})$ be fixed and let $k \in \{0, \dots, d\}$. We fix once and for all an $\mathrm{O}(k) \times \mathrm{O}(d-k)$-invariant submanifold

$$T_k \subset [0,1] \times D^k \times D^{d-k}$$

with the following properties (see Figure 1.7 for a visualization)

1. $(s, 0, 0) \in T_k$ if and only if $s = \frac{1}{2}$.
2. The projection $T_k \xrightarrow{pr} [0,1]$ is a Morse function and $(\frac{1}{2}, 0, 0)$ is the only critical point of this Morse function. Its index is k.
3. We have the following equalities for intersections

$$T_k \cap ([0,\varepsilon) \times S^{k-1} \times D^{d-k}) = [0,\varepsilon) \times S^{k-1} \times D^{d-k}$$
$$T_k \cap ((1-\varepsilon, 1] \times D^k \times S^{d-k-1}) = (1-\varepsilon, 1] \times D^k \times S^{d-k-1}$$
$$T_k \cap ([0,1] \times S^{k-1} \times S^{d-k-1}) = [0,1] \times S^{k-1} \times S^{d-k-1}$$

4. The boundary of T_k is given by

$$\partial T_k = (\{0\} \times S^{k-1} \times D^{d-k}) \cup (\{1\} \times D^k \times S^{d-k-1}) \cup ([0,1] \times S^{k-1} \times S^{d-k-1}).$$

We call T_k the *standard trace* of a k-surgery.

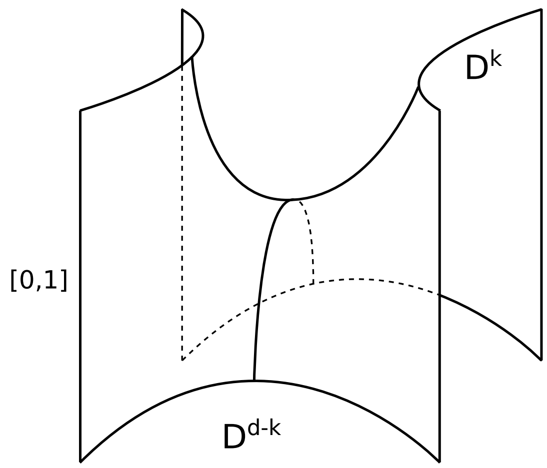


FIGURE 1.7: A standard trace.

Definition 1.5.2 (Trace of a surgery)**.** Let M be a manifold and let $\varphi\colon S^{k-1}\times D^{d-k} \hookrightarrow M$ be an embedding. We call such an embedding a *k-surgery datum in M* and we define the *trace of* φ to be

$$\textbf{tr}\,(\varphi) := \Big([0,1] \times (M \setminus \operatorname{im}\varphi)\Big) \cup_{\operatorname{id}_{[0,1]}\times\varphi} T_k.$$

There is a Morse function $h_\varphi\colon \textbf{tr}\,(\varphi) \to [0,1]$ with precisely one critical point with value $\frac{1}{2}$ and index k. We define $M_\varphi := h_\varphi^{-1}(1) \cong (M \setminus \operatorname{im}\varphi) \cup (D^k \times S^{d-k-1})$.

For a surgery datum φ in M there is an obvious reversed surgery datum $\varphi^{\mathrm{op}}\colon S^{d-k-1} \times D^k \hookrightarrow M_\varphi$ and there is a canonical diffeomorphism $(M_\varphi)_{\varphi^{\mathrm{op}}} \cong M$. We define the *attaching sphere of* φ to be $\varphi(S^{k-1} \times \{0\}) \subset M$ and the *belt sphere of* φ as $\varphi^{\mathrm{op}}(\{0\} \times S^{d-k-1}) \subset M_\varphi$.

Definition 1.5.3. 1. Let $W\colon M_0 \rightsquigarrow M_1$ be a cobordism and let $\varphi\colon S^{k-1} \times D^{d-k} \hookrightarrow M_1$ be an embedding. We define the manifold W *with a k-handle attached along* φ to be $W \cup \textbf{tr}\,(\varphi)$.

2. A *handle decomposition of* $W\colon M_0 \rightsquigarrow M_1$ is a collection of manifolds $N_1, \dots, N_n$, embeddings $\varphi_i\colon S^{k_i-1} \times D^{d-k_i} \hookrightarrow N_i$ for $i = 1, \dots, n$ and diffeomorphisms

$f_0 \colon M_0 \xrightarrow{\cong} N_1$, $f_i \colon (N_i)_{\varphi_i} \xrightarrow{\cong} N_{i+1}$ for $i = 1, \dots, n-1$ and $f_n \colon (N_n)_{\varphi_n} \xrightarrow{\cong} M_1$ such that there exists a diffeomorphism $\mathrm{rel}\ M_0, M_1$

$$W \cong M_0 \times [0,1] \cup_{f_0} \mathbf{tr}\,(\varphi_0) \cup_{f_1} \mathbf{tr}\,\varphi_1 \cup \cdots \cup \mathbf{tr}\,(\varphi_{n-1}) \cup_{f_n} M_1 \times [0,1].$$

We call f_i the *identifying diffeomorphisms* and φ_i the *surgery data*.

Remark 1.5.4. For a diffeomorphism $f \colon M_0 \xrightarrow{\cong} M_1$ and a surgery datum φ in M_0 there exists a canonical induced diffeomorphism $F \colon \mathbf{tr}\,\varphi \to \mathbf{tr}\,(f \circ \varphi)$ that restricts to f on the incoming boundary and to a diffeomorphism $f_\varphi \colon (M_0)_\varphi \to (M_1)_{f\circ\varphi}$ satisfying $f_\varphi \circ \varphi^{\mathrm{op}} = (f \circ \varphi)^{\mathrm{op}}$ on the outgoing boundary. Furthermore f_φ is equal to f on $M_0 \setminus \mathrm{im}\,\varphi$.

In order to compare different handle decompositions of a manifold, we will now describe the model for handle cancellation. Let $W \colon M_0 \rightsquigarrow M_1$ be a cobordism which has a handle decomposition with two handles[2]: Let $\varphi \colon S^{k-1} \times D^{d-k} \hookrightarrow M_0$ and $\varphi' \colon S^k \times D^{d-k-1} \hookrightarrow (M_0)_\varphi$ be two surgery data such that the belt sphere of φ and the attaching sphere of φ' intersect transversely in a single point. By [Wal16, Theorem 5.4.3] there exists an embedding of an disk $D^{d-1} \cong D \subset M_0$ such that $\mathrm{im}\,\varphi \subset D$ and $\mathrm{im}\,\varphi' \subset D_\varphi$. Therefore it suffices to have a closer look at handle cancellation on the sphere. Let $M_0 = D \cup D' = S^{d-1}$ where D' is another disk. Let $a \in S^{d-k-1}$ and $b \in S^k$ such that $\varphi^{\mathrm{op}}(0,a) = \varphi'(b,0)$ is the unique intersection point. Since the belt sphere of φ and the attaching sphere of φ' intersect transversally here, there is a disc $S^k_+ \subset S^k$ such that $\varphi'(S^k_+ \times \{0\}) = \varphi^{op}(D^k \times \{a\})$ after possibly changing the coordinates of D. Let $S^k_- := \overline{S^k \setminus S^k_+}$. Then $\varphi'(S^k_- \times \{0\}) \subset M \setminus \mathrm{im}\,\varphi$ (see Figure 1.8).

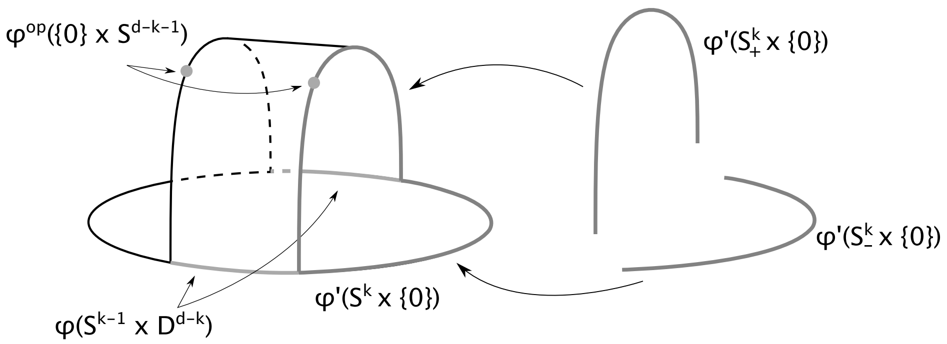


FIGURE 1.8

[2]For ease of notation we assume that $f_0 = \mathrm{id}_{M_0}$ and $f_1 = \mathrm{id}_{(M_0)_\varphi}$.

Because of transversality we may isotopy φ' such that $\varphi'(S^k_- \times D^{d-k-1}) \subset M \setminus \operatorname{im}\varphi$. Then $\varphi(S^{k-1} \times D^{d-k}) \cup \varphi'(S^k_- \times D^{d-k-1}) \cong D^{d-1}$ (cf. [Wal16, Lemma 5.4.2.]) and also $A := \overline{S^{d-1} \backslash \big(\varphi(S^{k-1} \times D^{d-k}) \cup \varphi'(S^k_- \times D^{d-k-1})\big)} \cong D^{d-1}$. By choosing an identification $A \cong D^k \times D^{d-k-1}$ we have $\varphi'(S^k_- \times D^{d-k-1}) \cup A \cong S^k \times D^{d-k-1}$. We see that

$$S^{d-1} = \big(\varphi(S^{k-1} \times D^{d-k}) \cup \underbrace{\varphi'(S^k_- \times D^{d-k-1})\big) \cup A}_{\cong S^k \times D^{d-k-1}}$$

and hence we can change coordinates on S^{d-1} by changing the embedding $D^{d-1} \hookrightarrow M$ such that φ is the embedding of the first factor of the solid torus decomposition $a^k \colon S^{d-1} \xrightarrow{\cong} (S^{k-1} \times D^{d-k}) \cup (D^k \times S^{d-k-1})$, i. e. $a_k \circ \varphi = \iota_{(S^{k-1} \times D^{d-k})}$. We get an induced map

$$a^k_\varphi \colon S^{d-1}_\varphi \xrightarrow{\cong} (D^k \times S^{d-k-1}) \cup (D^k \times S^{d-k-1}) = S^k \times S^{d-k-1} = (S^k \times D^{d-k-1}) \cup (S^k \times D^{d-k-1}).$$

Because of transversality we may isotope φ' so that $(a^k_\varphi) \circ \varphi'$ is equal to the inclusion of the first factor in $S^k \times D^{d-k-1} \cup S^k \times D^{d-k-1}$. Then

$$(a^k_\varphi)_{\varphi'} \colon (S^{d-1}_\varphi)_{\varphi'} \xrightarrow{\cong} D^{k+1} \times S^{d-k-2} \cup S^k \times D^{d-k-1}$$

This is a solid torus decomposition of $(S^{d-1}_\varphi)_{\varphi'}$. We get a diffeomorphism $H_k \colon S^{d-1} \times [0,2] \xrightarrow{\cong}$ **tr** $(\varphi) \cup$ **tr** (φ') which fixes the strip $D' \times [0,2] \subset (S^{d-1}) \times [0,2]$ and the lower boundary point-wise. We may also assume that H_k restricts on the upper boundary to a diffeomorphism $\eta_k \colon S^{d-1} \xrightarrow{\cong} (S^{d-1}_\varphi)_{\varphi'}$ which translates $(a^k_\varphi)_{\varphi'}$ into the solid torus decomposition a^{k+1}, i. e. $\big((a^k_\varphi)_{\varphi'} \circ \eta_k\big) = a^{k+1}$. For every $k \in \{0, \ldots, d\}$ we fix the diffeomorphisms H_k (and hence η_k) once and for all.

Next we will recall from [Mil65] how a Morse function gives rise to a handle decomposition.

Construction 1.5.5 ([Mil65, pp. 28]). Let $W \colon M_0 \rightsquigarrow M_1$ be a cobordism and let $h \colon W \to [0,1]$ be a Morse function with critical points $p_1, \ldots, p_n$ which have pairwise distinct values $c_1 < \cdots < c_n$ and indices $k_1, \ldots, k_n$. For $\varepsilon \in (0, \min_i(c_1, c_{i+1}-c_i, 1-c_n))$ we get[3] $W_i := h^{-1}[c_i-\varepsilon, c_i+\varepsilon]$, $Z_i := h^{-1}[c_i+\varepsilon, c_{i+1}-\varepsilon]$ so that $W = Z_0 \cup W_1 \cup Z_1 \cup \cdots \cup W_n \cup Z_n$. Let V be a gradient-like vector field for h, i. e. a vector field such that $V \cdot h > 0$ away from the critical points and near critical points it is equal to the gradient vector field of f with respect to some background metric on W. Now Z_i is a cobordism without critical point and following the flow-lines of V gives a diffeomorphism $\alpha_i \colon h^{-1}(c_{i-1} + \varepsilon) \xrightarrow{\cong}$

[3] Here we have the convention that $c_0 := -\varepsilon, c_{n+1} = 1 + \varepsilon$.

$h^{-1}(c_i-\varepsilon)$. Also we get a surgery embedding φ_i into $h^{-1}(c_i-\varepsilon)$ and a diffeomorphism $\beta_i\colon \mathbf{tr}\,\varphi_i \xrightarrow{\cong} W_i$ as follows: Let $g\colon D^d_\delta \to W$ be a Morse chart centered at p_i. For $(a,b) \in S^{k_i-1} \times D^{d-k_i}$ we define $\varphi_i(a,b)$ by taking $g(\frac{\delta}{2}a, \frac{\delta}{2}b)$ and following the flow-line of V until we reach $h^{-1}(c_i-\varepsilon)$. For $(s,p) \in [0,1] \times (h^{-1}(c_i-\varepsilon) \setminus \operatorname{im}\varphi)$, let $\alpha_i(s,p)$ be the unique point q lying on the flow-line of V through p which satisfies $h(p) = c_i - \varepsilon + 2s\varepsilon$ and for $(s,a,b) \in T_{k_i}$, the value $\beta_i(s,a,b)$ is similarly defined to be the unique point q lying on the flow-line[4] through $g(\frac{\delta}{2}a, \frac{\delta}{2}b)$ that also satisfies $h(p) = c_i - \varepsilon + 2s\varepsilon$.

Now we can give a handle decomposition of W relative to M_0: Let $N_i := h^{-1}(c_i-\varepsilon)$, φ_i as above and let $f_0 := \alpha_0$ and $f_i := \alpha_i \circ \beta_i$ for $i \geq 1$. This gives a handle decomposition of W.

Remark 1.5.6. If g' is a different coordinate neighbourhood of W at p_i, $g^{-1} \circ g'$ is a diffeomorphism of the disk which preserves $\langle x_1, \ldots, x_{k_i}\rangle$ and $\langle x_{k_i+1}, \ldots, x_d\rangle$ and hence it is isotopic to an element in $O(k) \times O(d-k)$. Since V is unique up to isotopy, a different choice of V gives isotopic surgery embeddings φ_i and identifying diffeomorphisms f_i. If h_t is a path of Morse functions with distinct critical points, the difference of handle decompositions for h_0 and h_1 is the following: Critical points can be moved around but their order cannot be changed nor can they be cancelled. This means that we get isotopic surgery data φ_i and isotopic diffeomorphisms f_i.

Proposition 1.5.7. *Any two handle decompositions of W only differ by a finite sequence of the following moves:*

1. *An identifying diffeomorphism is replaced by an isotopic one.*
2. *A surgery datum is replaced by an isotopic one.*
3. *A k-surgery datum is precomposed with an element $A \in \mathrm{O}(k) \times \mathrm{O}(d-k)$.*
4. *The order of two surgery data with disjoint images is changed.*
5. *Let φ and φ' are k- and $(k+1)$-surgery data such that the belt sphere of φ and the attaching sphere of φ' intersect transversally in a single point. Then the two handles are replaced by the identifying diffeomorphism* $\mathrm{id} \# \eta_k$.

Proof. First let h_0 and h_1 be *any* two Morse functions on $W\colon M_0 \rightsquigarrow M_1$ with the number of critical values denoted by m_0 and m_1, respectively. By an isotopy through Morse functions with distinct critical values, we may assume that the critical values

[4]If $(a,b) \neq (0,0)$, this flow-line is unique.

are $\frac{1}{2m_0}, \frac{3}{2m_0}, \ldots, \frac{2m_0-1}{2m_0}$ and $\frac{1}{2m_1}, \frac{3}{2m_1}, \ldots, \frac{2m_1-1}{2m_1}$, respectively. By Lemma A.2 there exists a generic path of generalized Morse functions connecting h_0 and h_1. Let $T = \{t_1, \ldots, t_k\} \subset [0,1]$ denote the set of singular times, fix $\varepsilon < \frac{1}{2}\min_{i=2,\ldots,k}\{t_1, t_i - t_{i-1}, 1 - t_k\}$ and let $T_\varepsilon := \cup_{i=1}^k (t_i - \varepsilon, t_i + \varepsilon)$. Outside of T_ε, h_t is a path of Morse functions with distinct critical values. This is described in Remark 1.5.6 and is covered by relations 1 and 2. By isotoping h_t we may further assume that it has the following property: If $t \notin T_\varepsilon$ and h_t has m critical values, then their values are $\frac{1}{2m}, \frac{3}{2m}, \ldots, \frac{2m-1}{2m}$. The effect of this isotopy on the Cerf-Kirby-graphic is shown in Figure 1.9.

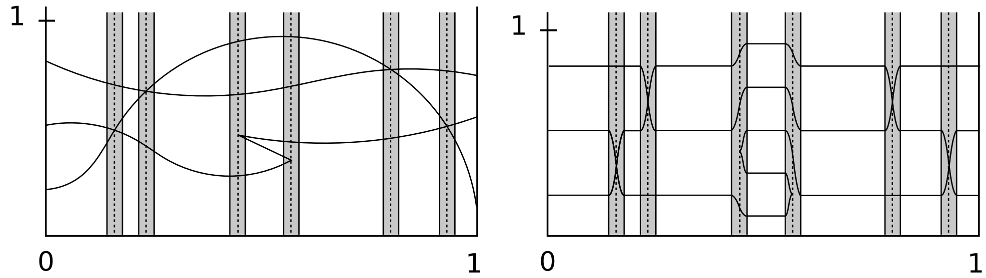


FIGURE 1.9: Left: The Cerf-Kirby-graphic associated to a generic path of generalized Morse functions. T_ε is the grey region.
Right: The Cerf-Kirby-graphic for the isotoped path of generalized Morse functions.

We now consider this problem locally in t, i. e. we only have to examine what happens at a singular time t_j. We may assume that $h_{t_j-\varepsilon}$ has either 2 critical points with value $\frac{1}{4}$ and $\frac{3}{4}$ or none at all. The same holds for $h_{t_j+\varepsilon}$. There are 2 possibilities: Either h_{t_j} has two critical points with the same value or it has a birth-death-singularity. In the former case this means that the critical values interchange (since critical values are isolated, we may assume the surgery data are disjoint). Now let p be a death-singularity (the case of a birth-singularity can be dealt with completely analogously). Since p is generically unfolded by h_t, by Lemma 1.4.9 there exists a $\delta > 0$ and a Morse chart $g\colon D^d \to W$ such that for $t - t_j < \delta$ we have

$$h_t(g(x_1, \ldots, x_d)) = x_1^3 - (t_j - t) \cdot x_1 + \sum_{i=2}^{k} -x_i^2 + \sum_{k+1}^{d} x_i^2 + \text{const}.$$

Now, for $t < t_j$ there are two critical points of $h_t \circ g$, namely $x_\pm := (\pm\sqrt{t_j - t}, 0, \ldots, 0)$ of index k and $k+1$ respectively. In order to get Morse charts $g_\pm$ for $x_\pm$ it suffices to change g in the first variable, i. e. there exist diffeomorphisms $\xi^t_\pm\colon \mathbb{R} \cong \mathbb{R}$ such that

$$h_t(\underbrace{g(\xi^t_\pm(x_1), x_2, \dots, x_d)}_{=:g^t_\pm(x_1,\dots,x_d)}) = \pm x_1^2 + h_t(g(0, x_2, \dots, x_d))$$

near $x_\pm$. Let φ^t_-, φ^t_+ be the respective surgery embeddings. The belt sphere of φ^t_- and the attaching sphere of φ^t_+ are given by

$$\varphi^t_-(x_1, \underbrace{x_2, \dots, x_{k-1}}_{=:v'}, 0, \dots, 0) = g^s_+(\frac{\delta}{2} \cdot x_1, \frac{\delta}{2} \cdot v', 0, \dots, 0)$$
$$\varphi^t_+(x_1, 0, \dots, 0, \underbrace{x_k, \dots, x_d}_{=:v}) = g^s_+(\frac{\delta}{2} \cdot x_1, 0, \dots, 0, \frac{\delta}{2} \cdot v)$$

and hence they intersect transversely in a single point. So there exists a disk $D^{d-1} \subset M_0$ such that $\operatorname{im} \varphi^{t_j-\varepsilon}_- \subset D^{d-1}$ and $\operatorname{im} \varphi^{t_j-\varepsilon}_+ \subset D^{d-1}_{\varphi^{t_j-\varepsilon}_-}$ and a diffeomorphism $W \cong$ **tr** $\varphi^{t_j-\varepsilon}_- \cup$ **tr** $\varphi^{t_j-\varepsilon}_+$. Since $h_{t_k+\varepsilon}$ has no critical points the two handles are cancelled and we get a diffeomorphism $\mathrm{id} \#_\partial H_k\colon M_0 \times [0,2] \xrightarrow{\cong}$ **tr** $\varphi^{t_j-\varepsilon}_- \cup$ **tr** $\varphi^{t_j-\varepsilon}_+$. □

Note that for any handle decomposition of W there exists a Morse function h on W such that Construction 1.5.5 yields this decomposition.

1.6 The 2-index theorem of Hatcher and Igusa

Definition 1.6.1. We define $\mathcal{H}(W)$ to be the space of generalized Morse functions on W. For $i \leq j \in \{0, \dots, d\}$ we denote by $\mathcal{H}_{i,j}(W)$ the space of generalized Morse functions such that non degenerate critical points have index in $\{i, \dots, j\}$ and A_2-singularities have index in $\{i, \dots, j-1\}$.

Let $W^d\colon M_0 \rightsquigarrow M_1$ be a cobordism. In this section we will prove the following theorem.

Theorem 1.6.2. *Let $d \geq 7$ and let $M_1 \hookrightarrow W$ be 2-connected. Then the space $\mathcal{H}_{0,d-3}(W)$ is path-connected. If furthermore $M_0 \hookrightarrow W$ is 2-connected as well, the space $\mathcal{H}_{3,d-3}(W)$ is path-connected. In particular, there exists a Morse-function without critical values of index $\{d-2, d-1, d\}$ or $\{0, 1, 2, d-2, d-1, d\}$ respectively.*

This follows from the *parametrized handle exchange theorem*. It was first proven by Hatcher [Hat75] "in a short and elegant paper which ignores most technical details"

[Igu88, p. 5]. A complete and rigorous proof has been given by Igusa in [Igu88]. Note that there is an index shift: Igusa considers $n+1$-dimensional cobordisms, whereas our cobordisms are d-dimensional.

Parametrized Handle Exchange Theorem ([Igu88, p. 211, Theorem 1.1])**.** *Let $i, j, k \in \mathbb{N}$ and assume that*

1. *(W, M_0) is i-connected,*
2. *$j \geq i+2$,*
3. *$i \leq d-k-2-\min(j-1, k-1)$,*
4. *$i \leq d-k-4$.*

Then the inclusion $\mathcal{H}_{i+1,j}(W) \hookrightarrow \mathcal{H}_{i,j}(W)$ is k-connected. There is a dual version of this: Assume that

1. *(W, M_1) is $d-j$-connected,*
2. *$j \geq i+2$,*
3. *$d-j \leq d-k-2-\min(j-1, k-1)$,*
4. *$d-j \leq d-k-4$.*

Then the inclusion $\mathcal{H}_{i,j-1}(W) \hookrightarrow \mathcal{H}_{i,j}(W)$ is k-connected.

Proof of Theorem 1.6.2. Consider the following chain of maps

$$\begin{array}{ccccccc} \mathcal{H}_{3,d-3}(W) & \longrightarrow & \mathcal{H}_{2,d-3}(W) & \longrightarrow & \mathcal{H}_{1,d-3}(W) & & \\ & & & & & \searrow & \\ & & & & & & \mathcal{H}_{0,d-3}(W) \\ & & & & & \swarrow & \\ \mathcal{H}(W) & \longleftarrow & \mathcal{H}_{0,d-1}(W) & \longleftarrow & \mathcal{H}_{0,d-2}(W) & & \end{array}$$

If $M_1 \hookrightarrow W$ is 2-connected and $d \geq 7$, the last three maps are 1-connected. If $M_0 \hookrightarrow W$ is 2-connected, the first three maps are 1-connected as well. The theorem follows as $\mathcal{H}(W)$ is connected. $\square$

Combining this with the work from the previous section we are able to compare different handle decomposition with index constraints on the surgery data.

Definition 1.6.3. Let $d \geq 7$. A cobordism $W = (W^d, \psi_0, \psi_1)\colon M_0 \rightsquigarrow M_1$ is called *admissible* if $\psi_1^{-1}\colon M_1 \hookrightarrow W$ is 2-connected. An *admissible handle decomposition* is a handle decomposition where all surgery data $\varphi_i\colon S^{k_i-1} \times D^{d-k_i} \hookrightarrow N_i$ satisfy $k_i \leq d-3$.

Proposition 1.6.4. *Let $W\colon M_0 \rightsquigarrow M_1$ be an admissible cobordism. Then any two admissible handle decompositions of W only differ by a finite sequence of the 5 moves from Proposition 1.5.7 with the following difference:*

5′. Let $k \leq d-4$ and let φ and φ' be k- and $(k+1)$-surgery data such that the belt sphere of φ and the attaching sphere of φ' intersect transversally in a single point. Then the two handles are replaced by the identifying diffeomorphism $\mathrm{id} \# \eta_k$.

Let us close this chapter by recalling the following Lemmas that make it possible to translate restrictions on the indices in a handle decomposition into conditions on tangential structures.

Lemma 1.6.5 ([Wal71, Theorem 3], see also [Sma62])**.** *Let $r \geq 0$ and let $d \geq \max\{r+4, s+4, r+s+2\}$. Let $W^d\colon M_0 \to M_1$ be a cobordism (of not necessarily closed manifolds M_i) such that (W, M_0) is r-connected and (W, M_1) is s-connected for $i = 0, 1$. Then there exists a handle decomposition of W without handles of indices $0, 1, \ldots, r, d-s, \ldots, d-1, d$.*

In particular, if $d \geq 6$ and (W, M_i) is 2-connected, there exists a handle decomposition of W without handles of indices $0, 1, 2, d-2, d-1, d$.

Remark 1.6.6.

1. In [Wal71] this Lemma is only stated for the case that (X, W_0) is 2-connected and $d \geq 5$. The proof however shows that the symmetrical statement above is true for $d \geq 6$ because in this case there are enough middle dimensions available.
2. This also follows from the Parametrized Handle Exchange Theorem for $k = 0$.

Lemma B.4 ([Kre99, Proposition 4], [HJ13, Proposition, Appendix III])**.** *Let $\theta\colon B \to BO(d)$ be a tangential structure, with B of type F_n. Let $W^d\colon M_0 \rightsquigarrow M_1$ be a θ-cobordism and let $M_1 \to B$ be n-connected. If $n \leq \frac{d}{2} - 1$, there exists a θ-cobordism $W'\colon M_0 \rightsquigarrow M_1$ such that (W', M_1) is d-connected. If $M_0 \to B$ is also n-connected, there exists a θ-cobordism $W'\colon M_0 \rightsquigarrow M_1$ such that (W', M_i) is n-connected for $i = 0, 1$. Furthermore W' is θ-bordant to W relative to the boundary.*

Remark 1.6.7. Again, this Lemma is only stated in [Kre99] and [HJ13] without the second half. This is why we decided to give a proof in the appendix.

1.7 The space $\mathcal{R}^+(M)$

In this section we introduce the space of metrics of positive scalar curvature and explain its topology. Afterwards we will state the famous surgery theorem of Gromov–Lawson–Schoen–Yau and its generalization of Chernysh along with a few implications thereof.

Definition 1.7.1. We denote by $\mathcal{R}(M)$ to be the set of all Riemannian metrics on M.

We want to endow the set $\mathcal{R}(M)$ with a topology. Let us recall the definition of the C^∞-topology (cf. [Hir76, pp.34]). Let M, N be smooth manifolds. Let $f\colon M \to N$ be a smooth map and let $\alpha\colon U \to \mathbb{R}^n$, $\beta\colon V \to \mathbb{R}^m$ be charts of M and N respectively. Let $K \subset U$ be compact such that $f(K) \subset V$ and let $\varepsilon > 0$. We define $\mathcal{N}(f; \varphi, \psi, K, \varepsilon)$ to be the set of smooth functions f' with $f'(K) \subset \psi(V)$ such that

$$\left\| \mathrm{d}^k(\psi \circ f \circ \varphi^{-1})(x) - \mathrm{d}^k(\psi \circ f' \circ \varphi^{-1})(x) \right\| \leq \varepsilon$$

for all $x \in K$ and all $k \geq 0$.

Definition 1.7.2. The *weak C^∞-topology* on $C^\infty(M, N)$ is the one which has the collection of sets $\mathcal{N}(f; \varphi, \psi, K, \varepsilon)$ as a subbasis.

Remark 1.7.3. For M compact, the weak C^∞-topology on $C^\infty(M, N)$ can be characterized as follows: A sequence of smooth functions f_n converges to a smooth function f if for all $k \geq 0$ the derivatives $f_n^{(k)}$ converge point-wise to $f^{(k)}$.

Since a Riemannian metric is a fiberwise scalar product, it can be thought of as a section of the bundle $\mathrm{Sym}^2(T^*M)$ of symmetric bilinear forms on M. So, $\mathcal{R}(M) \subset C^\infty(M, \mathrm{Sym}^2 T^*M)$ and $\mathcal{R}(M)$ becomes a topological space via the subspace topology.

Let (M, g) be a Riemannian manifold. We denote by $\mathbf{scal}_g \in C^\infty(M)$ the scalar curvature of g.

Definition 1.7.4. We define the *space of metrics of positive scalar curvature* $\mathcal{R}^+(M)$ to be the open subspace of $\mathcal{R}(M)$ which contains those metrics whose scalar curvature is strictly positive everywhere.

Definition 1.7.5. Let M and N be compact manifolds of dimension $d - 1 \geq 0$ and let $\phi\colon N \hookrightarrow M$ be an embedding. For a metric g on N, we define

$$\mathcal{R}^+(M, \phi; g) := \{h \in \mathcal{R}^+(M)\colon \phi^* h = g\}.$$

For $N = \coprod_{i=1}^n S^{k_i-1} \times D^{d-k_i}$ and $g = g_\circ^{k_i-1} + g_{tor}^{d-k_i}$ we write $\mathcal{R}^+(M,\phi) := \mathcal{R}^+(M,\phi;g)$. Here, $g_\circ$ denotes the round metric and g_{tor} a torpedo metric[5]. If there is no chance of confusion, we will omit the dimension of these metrics.

Now, let M^{d-1} be a compact manifold and let $\phi\colon N^{k-1} \hookrightarrow M$ be an embedding of a compact manifold N with trivial normal bundle into M. Let g_N be a metric on N such that $\mathbf{scal}(g_N + g_{tor}) > 0$.The following is the well-known Gromov–Lawson–Schoen–Yau surgery theorem (cf. [GL80] and [SY79]).

Theorem 1.7.6. $\mathcal{R}^+(M) \neq \emptyset \iff \mathcal{R}^+(M,\phi;g_N + g_{tor}) \neq \emptyset$.

Remark 1.7.7. The statement of this theorem can be strengthened to hold for a disconnected manifold N where components possibly have different dimensions: Let M be a $d-1$-manifold, and let N_i be closed manifolds of dimension $k_i - 1$, let g_{N_i} be metrics on N_i such that $\mathbf{scal}(g_{N_i} + g_{tor}) > 0$ and let $d - k_i \geq 3$ for all i. Let further $N := \coprod_{i=1}^n N_i \times D^{d-k_i}$, $g := \coprod_{i=1}^n g_{N_i} + g_{tor}$ and let $\phi\colon N \hookrightarrow M$ be an embedding. Then $\mathcal{R}^+(M) \neq \emptyset \iff \mathcal{R}^+(M,\phi;g) \neq \emptyset$.

In [GL80] Gromov–Lawson used this result to determine which simply connected non-Spin manifolds admit a metric of positive scalar curvature. Later, the Spin-case was solved by Stolz [Sto92].

There is the following generalization which is originally due to Chernysh [Che04b] and has been first published by Walsh [Wal13]. A detailed exposition of Chernysh's proof can be found in [EF18]. Let M, N, φ and g as in Remark 1.7.7.

Theorem 1.7.8 (Parametrized Surgery Theorem [Che04b, Theorem 1.1], [Wal13, Main Theorem])**.** *The map*

$$\mathcal{R}^+(M,\varphi;g) \hookrightarrow \mathcal{R}^+(M)$$

is a weak homotopy equivalence. In particular, if M_1 is obtained from M_0 by a surgery along $\varphi\colon S^{k-1} \times D^{d-k} \hookrightarrow M_0$ of index $k \leq d-3$ then there exists a zig-zag of maps

$$\mathcal{R}^+(M_0) \overset{\simeq}{\hookleftarrow} \mathcal{R}^+(M_0,\varphi) \xrightarrow{\cong} \mathcal{R}^+(M_1,\varphi^{\mathrm{op}}) \hookrightarrow \mathcal{R}^+(M_1).$$

If furthermore $k \geq 3$, the rightmost map in this composition is also a weak equivalence and we obtain a zig-zag of weak equivalences from $\mathcal{R}^+(M_0)$ to $\mathcal{R}^+(M_1)$.

[5] A torpedo metric on D^{d-k} is an $\mathrm{O}(d-k)$-invariant metric of positive scalar curvature that restricts to the round metric on the boundary. For precise definitions see [Che04b], [Wal11] or [EF18].

There is an application to cobordism theory:

Theorem 1.7.9 ([Wal13, Corollary 4.2], [EF18, Theorem 1.5])**.** *Let $d \geq 6$, let $\theta\colon B \to BO(d)$ be a fibration and let M_0, M_1 be θ-manifolds of $d-1$ such that the underlying structure map $l\colon M_i \to B$ is 2-connected for $i = 0, 1$. If M_0 and M_1 are θ-cobordant, then $\mathcal{R}^+(M_0) \simeq \mathcal{R}^+(M_1)$.*

Remark 1.7.10. The space $\mathcal{R}^+(M)$ is homotopy equivalent to a CW-complex (see [Pal66, Theorem 13]). By Whitehead's theorem, a weak homotopy equivalence of CW-complexes is an actual homotopy equivalence. Therefore we may assume that weak homotopy equivalences of $\mathcal{R}^+(M)$ have actual homotopy-inverses.

Sometimes, we need to work with manifolds with boundary. Let V be a manifold with boundary $\partial V =: N$ and let $c\colon N \times [0, \varepsilon) \hookrightarrow V$ be a collar of the boundary. We define $\mathcal{R}^+(V)^\varepsilon$ to be the space of all psc metrics on V such that $c^*g = h + dt^2$ for some $h \in \mathcal{R}(N)$. Since $\mathbf{scal}\, h + dt^2 = \mathbf{scal}\, h$, the scalar curvature of h is positive and we get a restriction map

$$\mathrm{res}\colon \mathcal{R}^+(V)^\varepsilon \to \mathcal{R}^+(N)$$

which is a quasifibration by [Che04a, Theorem 1.1] and a Serre-fibration by [EF18, Theorem 1.1].

Remark 1.7.11. If we would consider non-collared metrics, restricting to the boundary would not yield a psc-metric: D^3 carries a metric of positive scalar curvature and so does every 3-dimensional subspace. Embed a full torus $D^2 \times S^1$ in D^3. This has a psc metric but its boundary is a 2-torus which cannot admit one by the Gauss–Bonnet theorem.

By [BERW17, Lemma 2.1] the homotopy type of $\mathcal{R}^+(V)^\varepsilon$ does not depend on ε and in fact $\mathcal{R}^+(V)^\varepsilon \to \operatorname*{colim}_{\varepsilon \to 0} \mathcal{R}^+(V)^\varepsilon := \mathcal{R}^+(V)$ is a weak homotopy equivalence. For $h \in \mathcal{R}^+(N)$ we write $\mathcal{R}^+(V)_h := \mathrm{res}^{-1}(h)$ and if $V\colon M_0 \rightsquigarrow M_1$ is a cobordism, then we write $\mathcal{R}^+(V)_{h_0,h_1} := \mathrm{res}^{-1}(h_0 \amalg h_1)$.

As another application of the generalized surgery theorem, Walsh proved the following in his thesis.

Lemma 1.7.12 ([Wal11, Theorem 3.1])**.** *Let $(W, \psi_0, \psi_1)\colon M_0 \rightsquigarrow M_1$ be a cobordism such that ψ_1^{-1} is 2-connected and let $g_0 \in \mathcal{R}^+(M_0)$. Then, there exists a metric $G \in \mathcal{R}^+(W)$ extending $\psi_0^* g_0$.*

2

Decomposition of cobordisms

2.1 Cobordism categories

Geometric cobordism theory has had a great revival in the past 2 decades, starting with the proof of the Mumford-conjecture by Madsen–Weiss [MW07] and its generalization to higher dimensions by Galatius–Randal-Williams [GRW14]. In this section we recall the definition of the cobordism category $\mathcal{C}ob_d$ and we give a different model for $\pi_0(\mathcal{C}ob_d)$.

Definition 2.1.1 ([GRW10, Definition 3.7])**.** We define the *cobordism category* $\mathcal{C}ob_d$ as follows:

$$\mathbf{obj}_{\mathcal{C}\mathrm{ob_d}} \coloneqq \{M \colon M \subset \mathbb{R}^{\infty-1} \text{ is a closed } (d-1)\text{-dimensional manifold}\}$$
$$\mathbf{mor}_{\mathcal{C}\mathrm{ob_d}}(M_0, M_1) \coloneqq \{(W, t)\}$$

where $t \in \mathbb{R}_{>0}$ and $W \subset [0, t] \times \mathbb{R}^{\infty-1}$ is a compact d-manifold such that there exists an $\varepsilon > 0$ with $W \cap ([0, \varepsilon) \times \mathbb{R}^{\infty-1}) = [0, \varepsilon) \times M_0$ and $W \cap ((t-\varepsilon, t] \times \mathbb{R}^{\infty-1}) = (t-\varepsilon, t] \times M_1$. Composition is given by

$$(W', t') \circ (W, t) \coloneqq (W \cup (W' + t), t' + t)$$

where $+t$ means the obvious shift map. We turn this into a nonunital topological category by imposing the objects carry the discrete topology and morphism sets are topologized via

$$\mathbf{mor}_{\mathcal{C}\mathrm{ob_d}}(M_0, M_1) \cong \coprod_{[W]} E\mathrm{Diff}_\partial(W)/\mathrm{Diff}_\partial(W) \times \mathbb{R}_{>0}$$

where W runs over all diffeomorphism classes of compact collared d-cobordisms from M_0 to M_1. Here we use the model $E\mathrm{Diff}_\partial(W) := \mathrm{Emb}_\partial(W, [0,1] \times \mathbb{R}^{\infty-1})$ which denotes all embeddings of W which are fixed near the boundary.

Remark 2.1.2. This is essentially the same definition as in [GRW10] except for taking the discrete topology on objects. This however does not affect the homotopy type of their classifying spaces (see [ERW17b, Theorem 5.2 and p. 23]).

Definition 2.1.3. Let $\mathcal{C}$ be a topological category. We define $\pi_0(\mathcal{C})$ to be the category with objects $\pi_0(\mathbf{obj}_{\mathcal{C}})$ and morphisms $\pi_0(\mathbf{mor}_{\mathcal{C}})$.

Definition 2.1.4. Let $\mathcal{B}ord_d$ be the category given by:

$$\mathbf{obj}_{\mathcal{B}\mathrm{ord_d}} := \{M \text{ is a closed } (d-1)\text{-dimensional manifold}\}$$
$$\mathbf{mor}_{\mathcal{B}\mathrm{ord_d}}(M_0, M_1) := \{(W, \psi_0, \psi_1)\}/\sim$$

where W is a d-dimensional manifold with boundary $\partial W = \partial_0 W \coprod \partial_1 W$, the maps $\psi_i \colon \partial_i W \xrightarrow{\cong} M_i$, $i = 0, 1$ are diffeomorphisms and $(W, \psi_0, \psi_1) \sim (W', \psi_0', \psi_1')$ if there exists a diffeomorphism $F \colon W \to W'$ such that $\psi_i = \psi_i' \circ F|_{\partial_i W}$ for $i = 0, 1$. Composition is given by gluing

$$(W', \psi_0', \psi_1') \circ (W, \psi_0, \psi_1) = (W \cup_{(\psi_0')^{-1} \circ \psi_1} W', \psi_0, \psi_1'),$$

where for a diffeomorphism $f \colon \partial_1 W \xrightarrow{\cong} \partial_0 W'$ we denote by $W \cup_f W'$ the manifold obtained from $W \amalg W'$ by identifying $\partial_1 W$ and $\partial_0 W'$ along f.

Remark 2.1.5. In $\mathcal{B}ord_d$ the identity on M_0 is given by $(M_0 \times [0,1], \mathrm{id}, \mathrm{id})$.

Proposition 2.1.6. *The functor $\mathcal{F} \colon \pi_0(\mathcal{C}ob_d) \to \mathcal{B}ord_d$ given by $[W, t] \mapsto (W, \mathrm{id}, \mathrm{id})$ is an equivalence of categories.*

Proof. $\mathcal{F}$ is well-defined and faithful because (W, t) and (W', t') lie in the same component if and only if they are diffeomorphic relative to the boundary. $\mathcal{F}$ is essentially surjective by the Whitney embedding theorem and full because there is a diffeomorphism $(M_0 \times [0,1] \cup_{\psi_0^{-1}} W \cup_{\psi_1} M_1 \times [0,1], \mathrm{id}, \mathrm{id}) \xrightarrow{\cong} (W, \psi_0, \psi_1)$. □

Let Diff denote the groupoid with the same objects as $\mathcal{B}ord_d$ and morphism spaces given by $\mathbf{mor}_{\mathrm{Diff}}(M_0, M_1) = \mathrm{Diff}(M_0, M_1)$.

Proposition 2.1.7. *The map* $f \mapsto (M_0 \times [0,1], \mathrm{id}, f)$ *defines a functor* $\mathrm{Diff} \longrightarrow \mathcal{B}ord_d$.

Proof. Let $f \in \mathrm{Diff}(M_0, M_1)$ and $g \in \mathrm{Diff}(M_1, M_2)$. Then a diffeomorphism

$$\begin{aligned}(M_0 \times [0,2], \mathrm{id}, g \circ f) &\xrightarrow{\cong} (M_0 \times [0,1] \cup_f M_1 \times [1,2], \mathrm{id}, g) \\ &= (M_1 \times [1,2], \mathrm{id}, g) \circ (M_0 \times [0,1], \mathrm{id}, f)\end{aligned}$$

is given by $\mathrm{id}_{M_0 \times [0,1]} \cup (f \times \mathrm{id}_{[1,2]})$. □

2.2 The surgery datum category

We recall the following method to construct a category. For details see [Mac71, pp. 48].

Definition 2.2.1. A *graph* is a tupel $(O, A, \partial_0, \partial_1)$, where O and A are sets called *object set* and *arrow set* and ∂_0, ∂_1 are maps $A \rightrightarrows O$. We say that two arrows $f, g \in A$ are *composable* if $\partial_0 g = \partial_1 f$.

Definition 2.2.2. Let $G = (O, A, \partial_0, \partial_1)$ be a graph. We define the category $\mathcal{C}(G)$ to have elements of O as objects and morphisms of $\mathcal{C}(G)$ are (possibly empty) strings of composable morphisms of A. We call $\mathcal{C}(G)$ the *free category generated by G*.

Next we recall the notion of a *quotient category*.

Proposition 2.2.3 ([Mac71, p. 51, Proposition 1]). *1. Let $\mathcal{C}$ be a small category and let R be a binary relation, i. e. a map that assigns to each pair (a, b) of objects a subset of $\mathbf{mor}_{\mathcal{C}}(a, b)^2$. Then, there exists a category $\mathcal{C}/R$ with object set $\mathbf{obj}_{\mathcal{C}}$ and a functor $Q\colon \mathcal{C} \to \mathcal{C}/R$ (which is the identity on objects) such that*

(a) If $(f, f') \in R(a, b)$ then $Qf = Qf'$.

(b) If $H\colon \mathcal{C} \to \mathcal{D}$ is a functor such that $(f, f') \in R(a, b)$ implies $Hf = Hf'$, then there exists a unique functor $H'\colon \mathcal{C}/R \to \mathcal{D}$ such that $H' \circ Q = H$.

$\mathcal{C}/R$ is called the quotient category*. It is unique up to equivalence of categories.*

2. *We call R a* congruence *if for every pair of objects (a,b), the set $R(a,b)$ gives an equivalence relation and R respects composition, i.e. if $(f,f') \in R(a,b)$ and $g\colon a' \to a$, $h\colon b \to b'$, then $(hfg, hf'g) \in R(a',b')$. If R is a congruence, then morphism sets $\boldsymbol{mor}_{\mathcal{C}/\mathrm{R}}(a,b)$ of $\mathcal{C}/R$ are given by dividing out the equivalence relation $R(a,b)$ on $\boldsymbol{mor}_{\mathcal{C}/\mathrm{R}}(a,b)$.*

The main goal of this chapter is to give a presentation of $\mathcal{B}ord_d$, i.e. a graph G, a relation R and an equivalence of categories $\mathcal{C}(G)/R \xrightarrow{\cong} \mathcal{B}ord_d$. Let us first construct the graph G. Objects in O are the objects of $\mathcal{B}ord_d$ and arrows will be given by diffeomorphisms and elementary cobordisms:

1. For every diffeomorphism $f\colon M_0 \to M_1$ there is an arrow $I_f \in A$ connecting M_0 and M_1.
2. For every surgery datum φ in M there is an arrow $S_\varphi \in A$ connecting M and M_φ.

Next, we need to construct the relation R on $\mathcal{C}(G)$. Recall that for a diffeomorphism $f\colon M \to M'$ and a surgery datum φ in M there exists a canonical induced diffeomorphism $f_\varphi\colon M_\varphi \to M'_{f\circ\varphi}$. Also, if φ and φ' are two surgery embeddings into M with disjoint images, there is an induced obvious surgery embedding φ'_φ on M_φ and $(M_\varphi)_{\varphi'_\varphi} = (M_{\varphi'})_{\varphi_{\varphi'}}$. Now, let R be the relation on morphism sets of $\mathcal{C}(G)$ generated by the following:

1. $I_{\mathrm{id}} = \mathrm{id}$.
2. If $f\colon M_0 \xrightarrow{\cong} M_1$ and $g\colon M_1 \xrightarrow{\cong} M_2$ are diffeomorphisms, then $I_g \circ I_f = I_{g\circ f}$.
3. Let $f\colon M_0 \xrightarrow{\cong} M_1$ and let φ be a surgery embedding into M_0. Then $S_{f\circ\varphi} \circ I_f = I_{f_\varphi} \circ S_\varphi$.
4. If $f, f'\colon M \xrightarrow{\cong} M'$ are isotopic, then $I_f = I_{f'}$.
5. If $A \in O(k) \times O(d-k)$, then $S_\varphi = S_{\varphi\circ A}$.
6. If φ, φ' are two surgery embeddings into M with disjoint images, then $S_{\varphi_{\varphi'}} \circ S_{\varphi'} = S_{\varphi'_\varphi} \circ S_\varphi$.
7. Let φ be a k-surgery datum in M and φ' a $(k+1)$-surgery datum in M_φ such that the belt sphere of φ and the attaching sphere of φ' intersect transversely in a single point. Then $S_{\varphi'} \circ S_\varphi = I_{\mathrm{id}\,\#\,\eta_k}$, where η_k is the diffeomorphism described Section 1.5, below Remark 1.5.4.

Remark 2.2.4. For isotopic surgery embeddings φ and φ' we get a diffeotopy H of M such that $H_0 = id$ and $H_1 \circ \varphi = \varphi'$ by the isotopy extension theorem. Then

$$S_{\varphi'} = S_{H_1 \circ \varphi} \circ I_{H_0} = S_{H_1 \circ \varphi} \circ I_{H_1} = I_{(H_1)_\varphi} \circ S_\varphi.$$

Definition 2.2.5. We define the *surgery datum category* $\mathcal{X}_d$ to be $\mathcal{C}(G)/R$ and $Q\colon \mathcal{C}(G) \to \mathcal{X}_d$ shall denote the projection functor.

2.3 A presentation of the cobordism category

In this section we prove that the surgery datum gives a presentation of the category $\mathcal{B}ord_d$. This is the main result of this chapter.

Theorem 2.3.1. *Let $P\colon \mathcal{C}(G) \to \mathcal{B}ord_d$ denote the functor which is the identity on objects and is given on morphisms by*

1. *For $f\colon M_0 \to M_1$, I_f is mapped to $(M_0 \times [0,1], \mathrm{id}, f) \cong (M_1 \times [0,1], f^{-1}, \mathrm{id})$*
2. *For a surgery datum φ in M, S_φ is mapped to $(\mathbf{tr}\,(\varphi), \mathrm{id}, \mathrm{id})$.*

Then P descends to a functor $\mathcal{P}\colon \mathcal{X}_d \to \mathcal{B}ord_d$ which is an equivalence of categories.

Proof. First we check well-definedness. By Proposition 2.2.3 it suffices to show that P respects the relations of $\mathcal{X}_d$.

1. $(M_0 \times [0,1], \mathrm{id}, \mathrm{id})$ is the identity.
2. $(M_1 \times [0,1], \mathrm{id}, f) \circ (M_0 \times [0,1], \mathrm{id}, g) := (M_0 \times [0,1] \cup_g M_1 \times [0,1], \mathrm{id}, f)$
$$\xrightarrow{\cong} (M_0 \times [0,2], \mathrm{id}, f \circ g)$$
and the diffeomorphism is given by the identity on $M_0 \times [0,1]$ and by the map $(p,t) \mapsto (g^{-1}(p), t+1)$ for $(p,t) \in M_1 \times [0,1]$.
3. Let φ be a surgery embedding into M_0 and let $f\colon M_0 \xrightarrow{\cong} M_1$ be a diffeomorphism.

$$P(I_{f_\varphi} \circ S_\varphi) = (\mathbf{tr}\,(\varphi) \cup (M_0)_\varphi \times [0,1], \mathrm{id}, f_\varphi)$$
$$P(S_{f \circ \varphi} \circ I_f) = ([0,1] \times M_0 \cup_f \mathbf{tr}\,(f \circ \varphi), \mathrm{id}, \mathrm{id})$$

We will show that both of these are diffeomorphic to $X := (M_0 \times [0,1] \cup \mathbf{tr}\,\varphi \cup_{f_\varphi} (M_1)_{f \circ \varphi} \times [0,1], \mathrm{id}, \mathrm{id})$. The diffeomorphism $X \xrightarrow{\cong} P(I_{f_\varphi} \circ S_\varphi)$ is given by shrinking $M_0 \times [0,1] \cup \mathbf{tr}\,\varphi$ to $\mathbf{tr}\,\varphi$ and by $f_\varphi \times \mathrm{id}$ on $(M_0)_\varphi \times [0,1]$. Recall that there is a canonical diffeomorphism $F\colon \mathbf{tr}\,\varphi \xrightarrow{\cong} \mathbf{tr}\,(f \circ \varphi)$. The diffeomorphism $X \xrightarrow{\cong} P(S_{f\circ\varphi} \circ I_f)$ is given by the identity on $M_0 \times [0,1]$, F on $\mathbf{tr}\,(\varphi)$ and by shrinking the collar of $(M_1)_{f\circ\varphi}$.

4. Let $f_t\colon M_0 \xrightarrow{\cong} M_1$ be a diffeotopy. Then we get a diffeomorphism $F\colon ([0,1] \times M_0, \mathrm{id}, f_0) \xrightarrow{\cong} ([0,1] \times M_0, \mathrm{id}, f_1)$ given by $F(t,x) = f_t^{-1} \circ f_0(x)$.

5. For every $A \in \mathrm{O}(k) \times \mathrm{O}(d-k)$, $\varphi \circ A$ is just a reparametrization of φ and hence this does not change $\mathbf{tr}\,(\varphi)$ since the standard model was chosen to be $\mathrm{O}(k) \times \mathrm{O}(d-k)$-invariant (cf. Construction 1.5.1).

6. Let φ, φ' be surgery embeddings into M with disjoint images and let U, U' be disjoint neigbourhoods of $\mathrm{im}\,\varphi, \mathrm{im}\,\varphi'$ in M. Let $F\colon [0,2] \times M \xrightarrow{\cong} [0,2] \times M$ be a diffeomorphism such that

   (a) $F|_{[0,\frac{\varepsilon}{2}) \times M \cup (2-\frac{\varepsilon}{2},2] \times M} = \mathrm{id}$
   (b) $F(t,x) = (t+1, x)$ for $1 - \varepsilon_1 > t > \varepsilon_1$ and $x \in U$
   (c) $F(t,x) = (t-1, x)$ for $2 - \varepsilon_1 > t > 1 + \varepsilon_1$ and $x \in U'$

   Then, F induces a diffeomorphism $\overline{F}\colon \mathbf{tr}\,(\varphi) \cup \mathbf{tr}\,(\varphi'_\varphi) \cong \mathbf{tr}\,(\varphi') \cup \mathbf{tr}\,(\varphi_{\varphi'})$ which is the identity on a collar of the boundary.

7. This is precisely the situation discussed below Remark 1.5.6.

Therefore there is an essentially surjective functor $\mathcal{P}\colon \mathcal{X}_d \to \mathcal{B}ord_d$. Every cobordism admits a handle decomposition (see Construction 1.5.5) and hence this functor is full. It remains to show that it is faithful. This follows from Proposition 1.5.7: Any two preimages of a cobordism W under $\mathcal{P}$ only differ by a finite sequence of the seven relations of $\mathcal{X}_d$. □

Definition 2.3.2. Let $a, b \in \{-1, 0, 1, \dots\}$. We define:

1. We define $\mathcal{B}ord_d^{a,b} \subset \mathcal{B}ord_d$ to be the wide[1] subcategory defined by the following: $\mathbf{mor}_{\mathcal{B}\mathrm{ord}_\mathrm{d}^{\mathrm{a,b}}}(M_0, M_1)$ contains those morphisms (W, ψ_0, ψ_1) where $\psi_0^{-1}\colon M_0 \hookrightarrow W$ is a-connected and $\psi_1^{-1}\colon M_1 \hookrightarrow W$ is b-connected. Here (-1)-connected shall be the empty condition.

2. $G^{a,b}$ to be the graph with the same object set as G and morphisms as follows: For $f\colon M_0 \xrightarrow{\cong} M_1$ we have $I_f \in A$ connecting M_0 and M_1 and for every surgery

[1] A subcategory is called wide if it contains all objects.

embedding $\varphi\colon S^{k-1} \times D^{d-k} \hookrightarrow M$ with $k \in [a+1, d-b-1]$ we have $S_\varphi \in A$ connecting M and M_φ. Analogously to Definition 2.2.5, we define $\mathcal{X}_d^{a,b} := \mathcal{C}(G^{a,b})/R$.

Note that $\mathcal{B}ord_d^{a,b}$ is a category by the Blakers-Massey excision theorem [Die08, Theorem 6.4.1].

Theorem 2.3.3. *For $d \geq 7$, the functor $\mathcal{P}^{-1,2}\colon \mathcal{X}_d^{-1,2} \to \mathcal{B}ord_d^{-1,2}$ defined as in Theorem 2.3.1 is an equivalence of categories.*

Proof. The proof goes along the same lines as the proof of Theorem 2.3.1. For fullness we note that if the inclusions $\psi_1^{-1}\colon M_1 \hookrightarrow W$ is 2-connected respectively, there exists a Morse function with all indices $\leq d-3$ by Theorem 1.6.2. Faithfulness follows from Proposition 1.6.4. □

3

The surgery map

Having the presentation of the category $\mathcal{B}ord_d$ from the previous section at hand we can now turn to the scalar curvature part of the picture. We define and analyze the *surgery map*. This is our main tool for studying the action of the mapping class group on metrics of positive scalar curvature.

3.1 Definition of the surgery map

Recall that $\mathcal{C}(G^{a,b})$ is the free category corresponding to the surgery datum category $\mathcal{X}_d^{a,b}$. Also let hTop denote the homotopy category of spaces, i. e. the category with spaces as objects and whose morphisms are the homotopy classes of maps.

Definition 3.1.1. We define a functor

$$\overline{\mathcal{S}}\colon \mathcal{C}(G^{-1,2}) \longrightarrow \mathrm{hTop}$$

by the following:

1. $\overline{\mathcal{S}}(M) = \mathcal{R}^+(M)$.
2. For $f\colon M_0 \xrightarrow{\cong} M_1$ the morphism I_f is mapped to $[g \mapsto f_* g]$, where $f_* := (f^{-1})^*$.

3. For $\varphi\colon S^{k-1}\times D^{d-k}\hookrightarrow M$ with $k\le d-3$,

$$S_\varphi \mapsto [\mathcal{R}^+(M) \dashrightarrow \mathcal{R}^+(M,\varphi) \xrightarrow{\cong} \mathcal{R}^+(M_\varphi,\varphi^{\mathrm{op}}) \hookrightarrow \mathcal{R}^+(M_\varphi)],$$

where the first map in this chain is the homotopy inverse to the inclusion and the second one works as follows: For a metric $\tilde{g}$ on $M\setminus \operatorname{im}\varphi$, the metric $\tilde{g}\cup\varphi_*(g_\circ^{k-1}+g_{tor}^{d-k})$ is mapped to $\tilde{g}\cup(\varphi^{\mathrm{op}})_*(g_{tor}^{k}+g_\circ^{d-k-1})$.

We will abbreviate $\overline{\mathcal{S}}_f := \overline{\mathcal{S}}(I_f)$ and $\overline{\mathcal{S}}_\varphi := \overline{\mathcal{S}}(S_\varphi)$.

Remark 3.1.2. We have $\overline{\mathcal{S}}(\mathbf{mor}_{\mathcal{C}(\mathrm{G}^{2,2})}(M_0,M_1)) \subset \mathrm{h}\mathbf{Iso}(\mathcal{R}^+(M_0),\mathcal{R}^+(M_1))$, i.e. $\overline{\mathcal{S}}$ maps morphisms in $\mathcal{C}(G^{2,2})$ to (the homotopy classes of) homotopy equivalences. This follows from the Parametrized Surgery Theorem (cf. Theorem 1.7.8).

Lemma 3.1.3. *Then $\overline{\mathcal{S}}$ induces a well-defined functor $\mathcal{X}_d^{-1,2}\longrightarrow \mathrm{hTop}$.*

Proof. For $d\le 2$ the statement and the proof of this theorem is trivial since $\mathbf{mor}_{\mathcal{X}_\mathrm{d}^{-1,2}}$ is generated by diffeomorphisms and it suffices to note that isotopic diffeomorphisms induce homotopic maps. Therefore we may assume $d\ge 3$ throughout this proof.

Throughout this proof we will draw dashed arrows for maps that contain inverses of weak homotopy equivalences (cf. Remark 1.7.10).

We need to show that the relations R from Definition 2.2.5 do not change the homotopy class of $\overline{\mathcal{S}}(\alpha)$ for $\alpha\in\mathbf{mor}_{\mathcal{X}_\mathrm{d}^{-1,2}}(M_0,M_1)$. This is obvious for relations $1,2$ and 4. For relation 5 this is easy as well, because $g_\circ+g_{tor}$ is $O(k)\times O(d-k)$-invariant. Also, $S_{f\circ\varphi}\circ I_f$ and $I_{f_\varphi}\circ S_\varphi$ give homotopic maps because of the following homotopy-commutative diagram.

$$\begin{array}{ccccccc}
\mathcal{R}^+(M_0) & \hookleftarrow & \mathcal{R}^+(M_0,\varphi) & \longrightarrow & \mathcal{R}^+((M_0)_\varphi,\varphi^{\mathrm{op}}) & \hookrightarrow & \mathcal{R}^+((M_0)_\varphi) \\
\downarrow f_* & & \downarrow f_* & & \downarrow (f_\varphi)_* & & \downarrow (f_\varphi)_* \\
\mathcal{R}^+(M_1) & \hookleftarrow & \mathcal{R}^+(M_1,f\circ\varphi) & \longrightarrow & \mathcal{R}^+((M_1)_{f\circ\varphi},(f\circ\varphi)^{\mathrm{op}}) & \hookrightarrow & \mathcal{R}^+((M_1)_{f\circ\varphi})
\end{array}$$

For relation 6 let φ,φ' be two surgery embeddings into M with disjoint images. Then there are inclusions $\mathcal{R}^+(M,\varphi)\hookleftarrow\mathcal{R}^+(M,\varphi\amalg\varphi')\hookrightarrow\mathcal{R}^+(M,\varphi')$ and performing both surgery maps at the same time is the same as performing them one after another.

The hardest part of this proof is to show that handle cancellation does not alter the homotopy class of $\overline{\mathcal{S}}(\alpha)$. If $d = 3$ the only surgery data in $\mathbf{mor}_{\mathcal{X}_{\mathrm{d}}^{-1,2}}$ are of the form $S^{-1} \times D^3 \hookrightarrow M$. Hence there cannot be cancelling surgeries and we may assume that $d \geq 4$ from now on. Let φ, φ' be surgery data in M as in relation 7 and let $f := \mathrm{id}_M \# \eta_k$ where $\eta_k \colon S^{d-1} \xrightarrow{\cong} (S^{d-1}_\varphi)_{\varphi'}$ is the fixed diffeomorphism from Section 1.5. Note that in this case we have $k \leq d-4$ and $d \geq 4$. There exists an embedding of a disk $D^{d-1} \cong D \subset M$ such that $\mathrm{im}\,\varphi \subset D$ and $\mathrm{im}\,\varphi' \subset D_\varphi$. It suffices to show that the composition

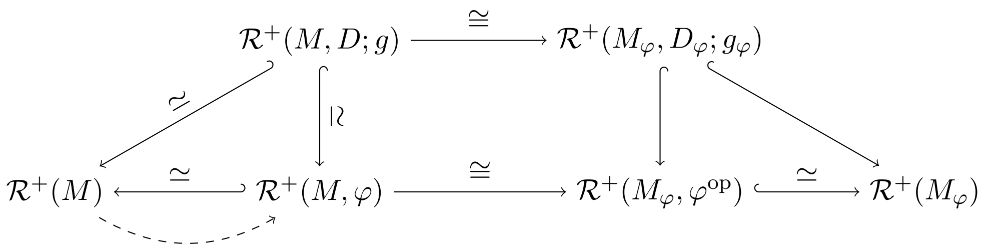
$$\mathcal{R}^+(M, D; g_{tor}) \xhookrightarrow{\iota} \mathcal{R}^+(M) \overset{\overline{\mathcal{S}}_{\varphi'} \circ \overline{\mathcal{S}}_\varphi}{\dashrightarrow} \mathcal{R}^+\big((M_\varphi)_{\varphi'}\big) \xrightarrow{f^*} \mathcal{R}^+(M)$$

is homotopic to the inclusion ι: Then by the Parametrized Surgery Theorem (cf. Theorem 1.7.8), the inclusion map ι is a weak homotopy equivalence since $d \geq 4$ and hence $\overline{\mathcal{S}}_{\varphi'} \circ \overline{\mathcal{S}}_\varphi$ is homotopic to f_*.

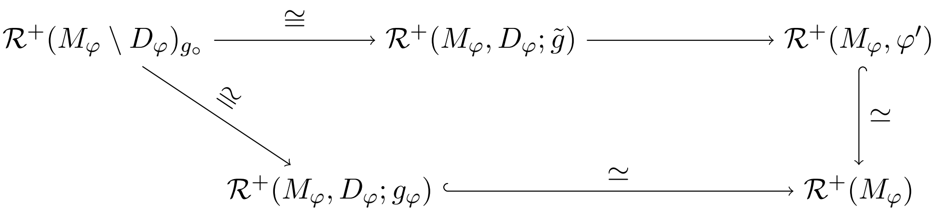
Let $g \in \mathcal{R}^+(D, \varphi)_{g_\circ}$ be a metric in the component of $g_{tor} \in \mathcal{R}^+(D)_{g_\circ}$. Consider the following diagram:

$$\begin{array}{ccccc}
\mathcal{R}^+(M \setminus D)_{g_\circ} & \xrightarrow{\cong} & \mathcal{R}^+(M, D; g) & \longrightarrow & \mathcal{R}^+(M, \varphi) \\
& \searrow^{\cong} & & & \big\downarrow \simeq \\
& & \mathcal{R}^+(M, D; g_{tor}) & \xhookrightarrow{\simeq} & \mathcal{R}^+(M)
\end{array}$$

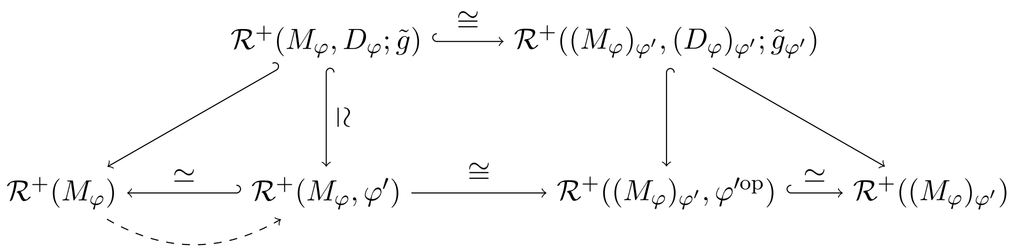
The composition of the top maps is given by gluing in g and the composition of the lower maps is given by gluing in g_{tor}. These two metrics are homotopic relative to the boundary and hence this diagram commutes up to homotopy. The bottom map and the right-hand vertical map are weak equivalences by the Parametrized Surgery Theorem (cf. Theorem 1.7.8) because $d \geq 4$ and $k \leq d-4$. Hence, the inclusion map $\mathcal{R}^+(M, D; g) \hookrightarrow \mathcal{R}^+(M, \varphi)$ is a weak equivalence as well. Let g_φ be the metric obtained from g by cutting out $\varphi_*(g_\circ^{k-1} + g_{tor}^{d-k})$ and gluing in $\varphi_*^{\mathrm{op}}(g_{tor}^k + g_\circ^{d-k-1})$. The following diagram commutes on the nose with the non-dashed arrows and up to homotopy with the dashed arrow:



It again follows that the right-hand vertical map and the right-hand diagonal map are weak equivalences. Note that the composition of the bottom horizontal maps is precisely the map $\overline{\mathcal{S}}_\varphi$. Now let $\tilde{g} \in \mathcal{R}^+(D_\varphi, \varphi')_{g_\circ}$ be a metric in the component of $g_\varphi \in \mathcal{R}^+(D_\varphi)_{g_\circ}$. We get the following diagram



which is homotopy commutative as $\tilde{g}$ and g_φ are homotopic. The righthand vertical map is a weak equivalence because $d-k-1 \geq 3$ and we deduce that $\mathcal{R}^+(M_\varphi, D_\varphi; \tilde{g}) \hookrightarrow \mathcal{R}^+(M_\varphi, \varphi')$ is a weak equivalence as well. Let $\tilde{g}_{\varphi'}$ be the metric obtained from $\tilde{g}$ by cutting out $\varphi'_*(g_\circ^k + g_{tor}^{d-k-1})$ and gluing in ${\varphi'^{\mathrm{op}}}_*(g_{tor}^{k+1} + g_\circ^{d-k-2})$. We get the analogous homotopy-commutative diagram:



This accumulates to the following diagram where all arrows are weak equivalences:

$$
\begin{array}{ccccccc}
\mathcal{R}^+(M,D;g_{tor}) & \longrightarrow & \mathcal{R}^+(M_\varphi,D_\varphi;g_\varphi) & \xrightarrow{(1)} & \mathcal{R}^+(M_\varphi,D_\varphi;\tilde{g}) & \longrightarrow & \mathcal{R}^+((M_\varphi)_{\varphi'},(D_\varphi)_{\varphi'};\tilde{g}_{\varphi'}) \\
\downarrow \iota & & \searrow & & \swarrow & & \downarrow \\
\mathcal{R}^+(M) & & \overset{\overline{\mathcal{S}}_\varphi}{\dashrightarrow} & \mathcal{R}^+(M_\varphi) & \overset{\overline{\mathcal{S}}'_\varphi}{\dashrightarrow} & & \mathcal{R}^+((M_\varphi)_{\varphi'}) \\
& & & & & & \downarrow f^* \\
& & & & & & \mathcal{R}^+(M)
\end{array}
$$

Here, the map (1) is given by cutting out g_φ and gluing in $\tilde{g}$. Since these are homotopic relative to the boundary, the inside triangle and hence the entire diagram commutes up to homotopy. Therefore, the composition $f^* \circ \overline{\mathcal{S}}_{\varphi'} \circ \overline{\mathcal{S}}_\varphi \circ \iota$ is homotopic to the inclusion if and only if the top row composition in this diagram is. In contrast to $f^* \circ \overline{\mathcal{S}}_{\varphi'} \circ \overline{\mathcal{S}}_\varphi \circ \iota$ this composition only consists of actual maps which are given as follows: For $h \in \mathcal{R}^+(M \setminus D)_{g_\circ}$ we have

$$
\begin{array}{ccccccc}
h \cup g_{tor} & \longmapsto & h \cup g_\varphi & \longmapsto & h \cup \tilde{g} & \longmapsto & h \cup \tilde{g}_{\varphi'} \\
& & & & & & \big\downarrow \\
& & & & & & h \cup f^*\tilde{g}_{\varphi'}
\end{array}
$$

We will denote the path component of a psc metric g on M by $[g] \in \pi_0(\mathcal{R}^+(M))$. By the above argument it suffices to show that $[f^*\tilde{g}_{\varphi'}] = [g_{tor}] \in \pi_0(\mathcal{R}^+(D)_{g_\circ})$. This is implied by Lemma 3.1.4 as follows: We can assume that $D \subset S^{d-1}$ is a hemisphere and we have $f^* \circ \overline{\mathcal{S}}_{\varphi'} \circ \overline{\mathcal{S}}_\varphi([g_{tor} \cup g_{tor}]) \sim [g_{tor} \cup f^*\tilde{g}_{\varphi'}]$ by the above argument for $M = S^{d-1}$ and $h = g_{tor}$. After possibly changing the coordinates of the disk D we may assume the following: If $a^k \colon S^{d-1} \xrightarrow{\cong} (S^{k-1} \times D^{d-k}) \cup (D^k \times S^{d-k-1})$ is the solid torus decomposition then $a^k \circ \varphi$ is given by the inclusion of the first factor and $a^k_\varphi \circ \varphi' \colon S^k \times D^{d-k-1} \hookrightarrow (S^k \times D^{d-k-1}) \cup (S^k \times D^{d-k-1})$ is also given by the inclusion of the first factor (cf. Section 1.5). In this case we have $f = \eta_k$. The metric $[g_{tor} \cup g_{tor}]$ is homotopic to the round metric by [Wal11, Lemma 1.9] and we have

$$
\begin{aligned}
[g_{tor} \cup f^*\tilde{g}_{\varphi'}] &\sim \eta_k^* \circ \overline{\mathcal{S}}_{\varphi'} \circ \overline{\mathcal{S}}_\varphi([g_{tor} \cup g_{tor}]) \sim \eta_k^* \circ \overline{\mathcal{S}}_{\varphi'} \circ \overline{\mathcal{S}}_\varphi([g_\circ]) \overset{\text{Lemma 3.1.4}}{\sim} [g_\circ] \\
&\sim [g_{tor} \cup g_{tor}].
\end{aligned}
$$

Also $g_1 := g_{tor} \cup f^*\tilde{g}_{\varphi'}$ and $g_2 := g_{tor} \cup g_{tor}$ are both in the image of the inclusion map $\mathcal{R}^+(D)_{g_\circ} \hookrightarrow \mathcal{R}^+(S^{d-1})$ which is a weak equivalence and since $[g_1] = [g_2]$ it follows that $[g_{tor}] = [f^*\tilde{g}_{\varphi'}] \in \pi_0(\mathcal{R}^+(D)_{g_\circ})$. □

Lemma 3.1.4. *Let $g_\circ \in \mathcal{R}^+(S^{d-1})$ be the round metric and let $a_k \colon S^{d-1} \xrightarrow{\cong} (S^{k-1} \times D^{d-k}) \cup (D^k \times S^{d-k-1})$ be the solid torus decomposition. Let $\varphi \colon S^{k-1} \times D^{d-k} \hookrightarrow S^{d-1}$ and let $\varphi' \colon S^k \times D^{d-k-1} \hookrightarrow S^{d-1}_\varphi$ be surgery data such that $a^k \circ \varphi$ and $a^k_\varphi \circ \varphi'$ are both given by the inclusion of the respective first factor. Then $\overline{\mathcal{S}}_{\varphi'} \circ \overline{\mathcal{S}}_\varphi([g_\circ]) \sim \overline{\mathcal{S}}_{\eta_k}([g_\circ]) = (\eta_k)_*[g_\circ]$.*

Proof. Let $g^k_{mtor} := (g^{k-1}_\circ + g^{d-k}_{tor}) \cup (g^k_{tor} + g^{d-k-1}_\circ)$ denote the *mixed torpedo metric* on$(S^{k-1} \times D^{d-k}) \cup (D^k \times S^{d-k-1})$. By [Wal11, Lemma 1.9]) we have $(a^k)^* g^k_{mtor} \sim g_\circ$ and hence

$$\begin{aligned}\overline{\mathcal{S}}_\varphi(g_\circ) &\sim \overline{\mathcal{S}}_\varphi\big((a^k)^* g^k_{mtor}\big) = \overline{\mathcal{S}}_\varphi(\overline{\mathcal{S}}_{(a^k)^{-1}}(g^k_{mtor}))\\ &\sim \overline{\mathcal{S}}_{(a^k_\varphi)^{-1}} \overline{\mathcal{S}}_{a^k\circ\varphi}(g_{mtor}) = (a^k_\varphi)^* \overline{\mathcal{S}}_{a^k\circ\varphi}(g^k_{mtor})\end{aligned}$$

Now $a^k \circ \varphi$ is given by the inclusion and hence

$$\overline{\mathcal{S}}_{a^k\circ\varphi}(g^k_{mtor}) \sim (g_{tor} + g_\circ) \cup (g_{tor} + g_\circ) \sim g_\circ + g_\circ \sim \underbrace{(g_\circ + g_{tor}) \cup (g_\circ + g_{tor})}_{=:\overline{g}}$$

on $(D^k \times S^{d-k-1}) \cup (D^k \times S^{d-k-1}) = S^k \times S^{d-k-1} = (S^k \times D^{d-k-1}) \cup (S^k \times D^{d-k-1})$. We can now compute

$$\begin{aligned}\overline{\mathcal{S}}_{\varphi'}\overline{\mathcal{S}}_\varphi(g_\circ) &\sim \overline{\mathcal{S}}_{\varphi'}\big((a^k_\varphi)^*\overline{g}\big) \sim (a^k_\varphi)_{\varphi'}{}^* \,\overline{\mathcal{S}}_{a^k_\varphi\circ\varphi'}(\overline{g})\\ &\sim (a^k_\varphi)_{\varphi'}{}^* \underbrace{\overline{\mathcal{S}}_{(a^k_\varphi)\circ\varphi'}\Big((g_\circ + g_{tor}) \cup (g_\circ + g_{tor})\Big)}_{=(g_{tor}+g_\circ)\cup(g_\circ+g_{tor})=g^{k+1}_{mtor}}\\ &\sim (a^k_\varphi)_{\varphi'}{}^* \, g^{k+1}_{mtor}\end{aligned}$$

We have to show that $(a^k_\varphi)_{\varphi'}{}^* \, g^{k+1}_{mtor} \sim \eta_{k*} g_\circ$ which is equivalent to $\eta^*_k \left((a^k_\varphi)_{\varphi'}\right)^* \, g^{k+1}_{mtor} \sim g_\circ$. But η_k was chosen such that $\left((a^k_\varphi)_{\varphi'} \circ \eta_k\right) = a^{k+1}$ and therefore

$$\eta^*_k (a^k_\varphi)_{\varphi'}{}^* g^{k+1}_{mtor} = (a^{k+1})^* g^{k+1}_{mtor} \sim g_\circ.$$ □

We get the following Corollary which follows immediately from Lemma 3.1.3 and Theorem 2.3.3.

Corollary 3.1.5. *Let $d \geq 7$. Then there is a unique functor*[1]

$$\overline{\mathcal{S}}\colon \mathcal{B}ord_d^{-1,2} \longrightarrow \mathrm{hTop}$$

which satisfies:

1. $\overline{\mathcal{S}}(M) = \mathcal{R}^+(M)$
2. $\overline{\mathcal{S}}_{(M\times I,\mathrm{id},f)} = f_*$
3. $\overline{\mathcal{S}}_{(\boldsymbol{tr}\,\varphi,\mathrm{id},\mathrm{id})}(g) = \overline{\mathcal{S}}_{\varphi}$.

Corollary 3.1.6. *Let $W = (W, \psi_0, \psi_1)\colon M_0 \rightsquigarrow M_1$ be an admissible cobordism. Then there is a well defined homotopy class of a map $\overline{\mathcal{S}}_W : \mathcal{R}^+(M_0) \to \mathcal{R}^+(M_1)$. If $W^{\mathrm{op}} := (W^{\mathrm{op}}, \psi_1, \psi_0)$ is also admissible, i.e. $\psi_0^{-1}\colon M_0 \hookrightarrow W$ is also 2-connected, then $\overline{\mathcal{S}}_W$ is a homotopy equivalence and a homotopy-inverse is given by $\mathcal{S}_{W^{\mathrm{op}}}$.*

Remark 3.1.7. The construction from the proof of [Wal11, Theorem 3.1] (cf. Lemma 1.7.12) show the following: If $W = (W, \mathrm{id}, \mathrm{id})\colon M_0 \rightsquigarrow M_1$ be an admissible cobordism, $g_0 \in \mathcal{R}^+(M_0)$ and $g_1 \in \mathcal{R}^+(M_1)$ are metrics such that $\overline{\mathcal{S}}_W([g_0]) \sim [g_1]$, then there exists a metric $G \in \mathcal{R}^+(W)_{g_0,g_1}$.

3.2 Surgery invariance of $\overline{\mathcal{S}}$

In this section we prove the following Lemma.

Lemma 3.2.1. *Let $d \geq 7$ and let M_0, M_1 be two $(d-1)$-manifolds, let $W = [W, \mathrm{id}, \mathrm{id}] \in$ **mor**$_{\mathcal{B}\mathrm{ord}_\mathrm{d}^{-1,2}}(M_0, M_1)$ and let $\Phi\colon S^{k-1} \times D^{d-k+1} \hookrightarrow \mathrm{Int}\, W$ be an embedding with $3 \leq k \leq d-3$. Then $\overline{\mathcal{S}}_W \sim \overline{\mathcal{S}}_{W_\Phi}$.*

Proof. First we note that for $3 \leq k \leq d-3$, W_Φ is again an admissible cobordism: Let $W^\circ := W \setminus \mathrm{im}\, \Phi$. Then $W^\circ \hookrightarrow W$ is $(d-k)$-connected and $W^\circ \hookrightarrow W_\Phi$ is $(k-1)$-connected by Lemma B.3. We have the following diagram:

$$\begin{array}{ccccc} W & \xleftarrow{(d-k)\text{-connected}} & W^\circ & \xrightarrow{(k-1)\text{-connected}} & W_\Phi \\ & \nwarrow_{2\text{-connected}} & \uparrow & \nearrow & \\ & & M_1 & & \end{array}$$

[1] By abuse of notation, we call this functor $\overline{\mathcal{S}}$ again.

Since $3 \leq k \leq d-3$, the inclusions $M_1 \hookrightarrow W^\circ$ and $M_1 \hookrightarrow W_\Phi$ are 2-connected and hence W_Φ is admissible.

We first prove Lemma 3.2.1 in the case that $k \neq 3$. Let $c\colon M_1 \times [1-\varepsilon, 1] \hookrightarrow W$ be a collar which does not intersect $\operatorname{im}\Phi$ and let $\gamma\colon [0,1] \times D^{d-1} \hookrightarrow W$ be an embedded, thickened path connecting $M_1 \times \{1-\varepsilon\}$ and $\operatorname{im}\Phi$. Let

$$\begin{aligned} W_1 &:= \operatorname{im} c \,\#_\partial \operatorname{im}\Phi := \operatorname{im} c \cup \operatorname{im}\gamma \cup \operatorname{im}\Phi \\ W_1' &:= \operatorname{im} c \,\#_\partial \operatorname{im}\Phi^{\mathrm{op}} \\ W_0 &:= W \setminus W_1. \end{aligned}$$

We choose γ, so that the boundaries of all of these are smooth. Then $W_1 \simeq M_1 \vee S^{k-1}$, $W_1' \simeq M_1 \vee S^{d-k}$, $W_0 \cup W_1 = W$ and $W_0 \cup W_1' = W_\Phi$. Since $M_1 \hookrightarrow W$ and $M_1 \hookrightarrow W_\Phi$ are 2-connected and $4 \leq k \leq d-3$, the maps $M_1 \vee S^{k-1} \simeq W_1 \hookrightarrow W$ and $M_1 \vee S^{d-k} \simeq W' \hookrightarrow W_\Phi$ are 2-connected as well.

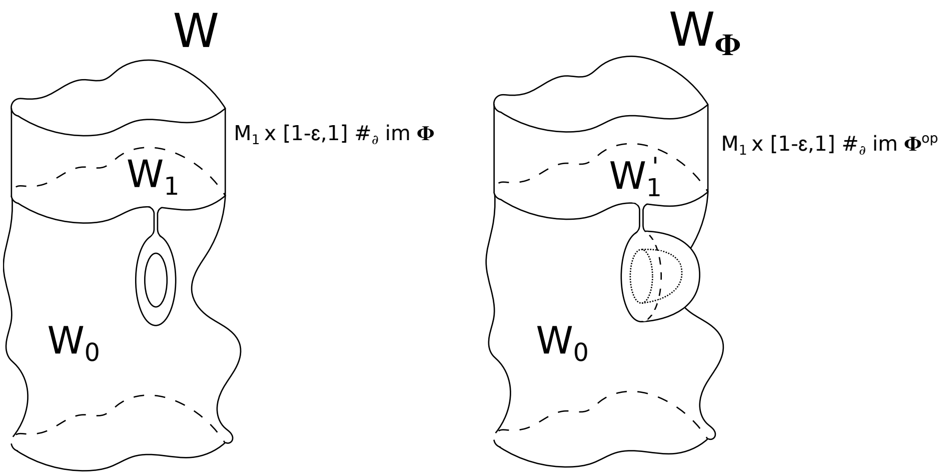


FIGURE 3.1: Surgery on the cobordism W

Note that W_1 and W_1' have the same boundary $M_1' \cong M_1 \# (S^{k-1} \times S^{d-k})$, namely

$$\partial W_1 = M_1 \amalg \underbrace{(M_1 \# \partial(\operatorname{im}\Phi))}_{=:M_1'} = M_1 \amalg (M_1 \# \partial(\operatorname{im}\Phi^{\mathrm{op}})) = \partial W_1'.$$

Next, we show that W_0, W_1, W_1' and W_1^{op} are again admissible. Becaus of $W_1 \simeq M_1 \vee S^{k-1}$ and $W_1' \simeq M_1 \vee S^{d-k}$:

- (W_1, M_1) is $(k-2)$-connected.

- (W_1, M_1') is $(d-k)$-connected.
- (W_1', M_1) is $(d-k-1)$-connected.

So, for $4 \leq k \leq d-3$ all of these are at least 2-connected and hence W_1, W_1' and W_1^{op} are admissible[2]. For W_0 we note that W is homotopy equivalent to W_0 with a $(d-k+1)$-cell attached along $\Phi(\{1\} \times S^{d-k})$:

$$\begin{aligned} W_0 \cup D^{d-k+1} &= \big(W \setminus (\mathrm{im}\ \Phi \cup \mathrm{im}\ \gamma)\big) \cup D^{d-k+1} \\ &= W \setminus (\underbrace{\mathrm{im}\ \Phi \setminus D^{d-k+1}}_{\simeq D^d} \cup \mathrm{im}\ \gamma) \simeq W. \end{aligned}$$

Therefore $W_0 \hookrightarrow W$ is $(d-k)$-connected and we have the following diagram.

$$\begin{array}{ccc} W_0 & \xrightarrow{(d-k)\text{-connected}} & W \\ \uparrow & & \uparrow \text{2-connected} \\ M_1' & \xrightarrow[\text{2-connected}]{} & W_1 \end{array}$$

and hence $M_1' \hookrightarrow W_0$ is 2-connected, too.

So we get a decompositions into admissible cobordisms $W = W_0 \cup W_1$ and $W_\Phi = W_0 \cup W_1'$ which implies $\overline{\mathcal{S}}_W = \overline{\mathcal{S}}_{W_1} \circ \overline{\mathcal{S}}_{W_0}$ and $\overline{\mathcal{S}}_{W_\Phi} = \overline{\mathcal{S}}_{W_1'} \circ \overline{\mathcal{S}}_{W_0}$. In the homotopy category hTop we have

$$\begin{aligned} \overline{\mathcal{S}}_{W_\Phi} &= \overline{\mathcal{S}}_{W_1'} \circ \underbrace{\overline{\mathcal{S}}_{W_1 \cup W_1^{\mathrm{op}}}}_{=\mathrm{id}} \circ \overline{\mathcal{S}}_{W_0} \\ &= \overline{\mathcal{S}}_{W_1'} \circ \overline{\mathcal{S}}_{W_1^{\mathrm{op}}} \circ \overline{\mathcal{S}}_{W_1} \circ \overline{\mathcal{S}}_{W_0} = \overline{\mathcal{S}}_{W_1^{\mathrm{op}} \cup W_1'} \circ \overline{\mathcal{S}}_W \end{aligned}$$

and so it suffices show that $W_1^{\mathrm{op}} \cup W_1'$ is diffeomorphic to $M_1 \times I$ relative to the boundary since $\overline{\mathcal{S}}_W$ only depends on the diffeomorphism type of W (see Lemma 3.1.3

[2]Note that without $k \geq 4$, the cobordism W_1 might not be admissible.

and its Corollary 3.1.5). We have (see Figure 3.2)

$$\begin{aligned} W_1^{\mathrm{op}} \cup W_1' &= \Big((M_1 \times [0,\varepsilon]) \#_\partial S^{k-1} \times D^{d-k+1}\Big) \\ &\quad \bigcup_{M_1'} \Big(D^k \times S^{d-k} \#_\partial (M_1 \times [1-\varepsilon, 1])\Big) \\ &\cong M_1 \times [0, 2\varepsilon] \# \underbrace{\Big((S^{k-1} \times D^{d-k+1}) \bigcup_{S^{k-1}\times S^{d-k}} (D^k \times S^{d-k})\Big)}_{\cong S^d} \\ &\cong M_1 \times [0,1]. \end{aligned}$$

and these diffeomorphisms are supported on a small neighbourhood of M_1' and hence relative to the boundary. This finishes the proof for the case $k \neq 3$.

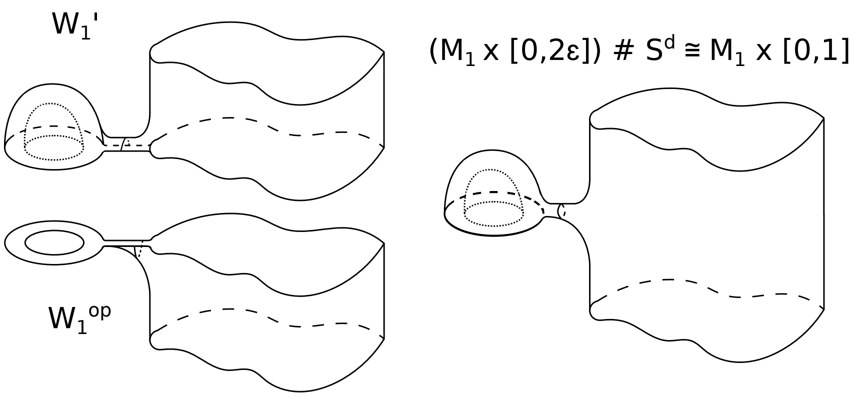


FIGURE 3.2: Gluing W_1^{op} to W_1'

For the case $k = 3$ we need a different argument, because W_1 might not be admissible in this case. Consider the map

$$\mathrm{Emb}(S^2 \times D^{d-3}, M_1) \longrightarrow \mathrm{Emb}(S^2 \times D^{d-2}, M_1 \times [0,2])$$

which is given by $\varphi \mapsto \Phi$ with $\Phi(x,(y,t)) = (\varphi(x,y),t)$ for $x \in S^2$ and $(y,t) \in D^{d-2} \subset D^{d-3} \times [0,1]$. We also have a map $\mathrm{Emb}(S^2 \times D^{d-2}, M_1 \times [0,2]) \hookrightarrow \mathrm{Emb}(S^2 \times D^{d-2}, W)$ given by shrinking the interval and composing with the inclusion of the collar. We will use the following Lemma.

Lemma 3.2.2. *In the present situation, the maps* $\mathrm{Emb}(S^2 \times D^{d-3}, M_1) \longrightarrow \mathrm{Emb}(S^2 \times D^{d-2}, M_1 \times [0,2])$ *and* $\mathrm{Emb}(S^2 \times D^{d-2}, M_1 \times [0,2]) \hookrightarrow \mathrm{Emb}(S^2 \times D^{d-2}, W)$ *are both* 0*-connected.*

By this Lemma we may isotope the embedding $\Phi\colon S^2 \times D^{d-2} \hookrightarrow W$ so that its image is contained in the collar of the boundary M_1. So we may assume that $W = M_1 \times [0,2]$. We abbreviate $M := M_1$. Again by the above lemma, we can isotope Φ such that $\Phi(S^2 \times D^{d-3} \times \{0\}) \subset M \times \{1\}$, i. e. Φ is a thickening of $\Phi|_{S^2\times D^{d-3}\times\{0\}}$. We abbreviate $\varphi := \Phi|_{S^2\times D^{d-3}\times\{0\}}$. Let us now give a diffeomorphism

$$\underbrace{(M \times [0,\tfrac{1}{2}] \cup_{\varphi} D^3 \times D^{d-3})}_{\cong \mathbf{tr}\,\varphi} \cup \underbrace{(M \times [\tfrac{1}{2},1] \cup_{\varphi} D^3 \times D^{d-3})}_{\cong (\mathbf{tr}\,\varphi)^{\mathrm{op}}} \xrightarrow{\alpha} \underbrace{(M \times I) \setminus \operatorname{im}\Phi \cup D^3 \times S^{d-3}}_{\cong (M\times I)_\Phi}.$$

On $(M \setminus \operatorname{im}\varphi) \times I$ the diffeomorphism α shall be given by the identity. Next we take diffeomorphisms

$$\alpha_1\colon \operatorname{im}\varphi \times [0,\frac{1}{2}] \xrightarrow{\cong} (\operatorname{im}\varphi \times [0,\frac{1}{2}]) \setminus (\operatorname{im}\Phi \cap [0,\frac{1}{2}])$$
$$\alpha_2\colon \operatorname{im}\varphi \times [\frac{1}{2},1] \xrightarrow{\cong} (\operatorname{im}\varphi \times [\frac{1}{2},1]) \setminus (\operatorname{im}\Phi \cap [\frac{1}{2},1]).$$

On the $D^3 \times D^{d-3}$-parts it is given by the inclusion of the lower or upper hemisphere $D^3 \times S^{d-3}_{\pm} \subset D^3 \times S^{d-3}$. This diffeomorphism is visualized in Figure 3.3. Therefore we have $\overline{\mathcal{S}}_{(M\times I)_\Phi} \sim \overline{\mathcal{S}}_{\mathbf{tr}\,\varphi^{\mathrm{op}}} \circ \overline{\mathcal{S}}_{\mathbf{tr}\,\varphi} \sim \mathrm{id} \sim \overline{\mathcal{S}}_{M\times I}$ and the proof is finished modulo Lemma 3.2.2. □

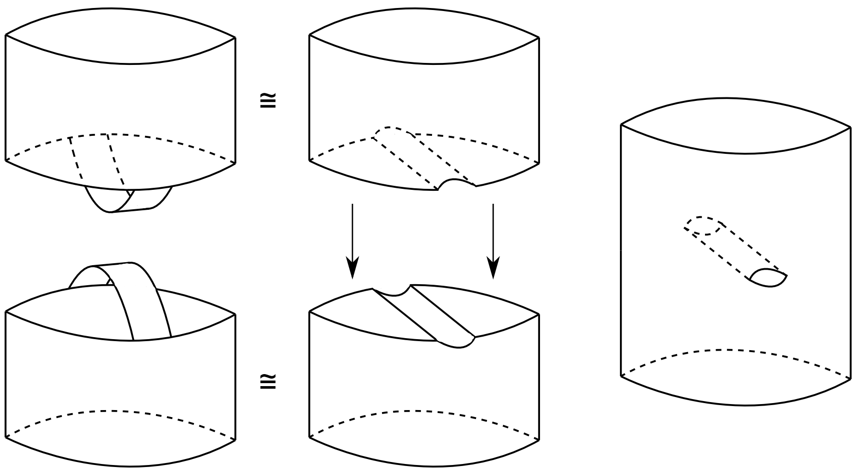


FIGURE 3.3: The diffeomorphism α.

Proof of Lemma 3.2.2. We have the following diagram

$$\begin{array}{ccccc}
\mathrm{Emb}(S^2 \times D^{d-3}, M_1) & \xrightarrow{(4)} & \mathrm{Emb}(S^2 \times D^{d-2}, M_1 \times [0,2]) & \xrightarrow{(5)} & \mathrm{Emb}(S^2 \times D^{d-2}, W) \\
\downarrow (1) & & & & \downarrow (6) \\
\mathrm{Imm}(S^2 \times D^{d-3}, M_1) & & & & \mathrm{Imm}(S^2 \times D^{d-2}, W) \\
\downarrow \simeq & & & & \downarrow \simeq \\
\mathrm{Mon}(TS^2 \oplus \underline{\mathbb{R}}^{d-3}, TM_1) & & & & \mathrm{Mon}(TS^2 \oplus \underline{\mathbb{R}}^{d-2}, TW) \\
\downarrow \cong & & & & \downarrow \cong \\
\mathrm{Map}(S^2, Fr(TM_1)) & \xrightarrow{(2)} & \mathrm{Map}(S^2, Fr(TM_1 \oplus \mathbb{R})) & \xrightarrow{(3)} & \mathrm{Map}(S^2, Fr(W))
\end{array}$$

where Mon denotes the space of bundle monomorphisms. Note that the bottom-most vertical maps are homeomorphisms because S^2 is stably parallelizable and the middle ones are homotopy equivalences by the Smale-Hirsch immersion theorem (cf. [Ada93, Section 3.9]). The map (1) is 0-connected because of the Whitney embedding (cf. [Hir76, pp. 26]) and the maps (5) and (6) are π_0-bijections by Lemma A.1. It remains to show that (2) and (3) are 0-connected. Then the map (4) is 0-connnected, too. For (2) consider the following diagram of fibrations.

$$\begin{array}{ccc}
\mathrm{Map}(S^2, \mathrm{Gl}_{d-1}(\mathbb{R})) & \xrightarrow{d-4\text{-conn.}} & \mathrm{Map}(S^2, \mathrm{Gl}_d(\mathbb{R})) \\
\downarrow & & \downarrow \\
\mathrm{Map}(S^2, \mathrm{Fr}(TM_1)) & \longrightarrow & \mathrm{Map}(S^2, \mathrm{Fr}(TM_1 \oplus \underline{\mathbb{R}})) \\
\downarrow & & \downarrow \\
\mathrm{Map}(S^2, M) & = & \mathrm{Map}(S^2, M)
\end{array}$$

Since $d - 4 \geq 3$, the map (2) is 0-connected. The map (3) fits in a similar diagram:

$$\begin{array}{ccc}
\mathrm{Map}(S^2, \mathrm{Gl}_d(\mathbb{R})) & = & \mathrm{Map}(S^2, \mathrm{Gl}_d(\mathbb{R})) \\
\downarrow & & \downarrow \\
\mathrm{Map}(S^2, \mathrm{Fr}(TM_1 \oplus \underline{\mathbb{R}})) & \longrightarrow & \mathrm{Map}(S^2, \mathrm{Fr}(W)) \\
\downarrow & & \downarrow \\
\mathrm{Map}(S^2, M_1) & \longrightarrow & \mathrm{Map}(S^2, W)
\end{array}$$

Since $M_1 \hookrightarrow W$ is 2-connected, the bottom-most map is 0-connected and hence so is the map (3). □

3.3 The factorization of the action map

In this section we state and prove the main theorem of this chapter and in fact the main result of this thesis. Before we can do so let us introduce some notation. Let $\theta\colon B \to BO(d)$ be a once-stable tangential structure. Let $\hat{\Omega}^\theta_{d,2}$ denote the category which has $(d-1)$-dimensional θ-manifolds $(M^{d-1}, \hat{l})$ as objects and the set of morphisms from $(M_0, \hat{l}_0)$ to $(M_1, \hat{l}_1)$ is given by $\Omega^\theta_d((M_0, \hat{l}_0), (M_1, \hat{l}_1))$ if the underlying structure map $l_1\colon M_1 \to B$ is 2-connected and by the empty set otherwise.

Theorem 3.3.1. *Let $d \geq 7$. Then there is a unique functor $\mathcal{S}\colon \hat{\Omega}^\theta_{d,2} \longrightarrow \mathrm{hTop}$ such that*

1. *$\mathcal{S}(M) = \mathcal{R}^+(M)$,*
2. *$\mathcal{S}(M \times I, \mathrm{id}, f^{-1}) = [g \mapsto f^*g]$,*
3. *$\mathcal{S}(\boldsymbol{tr}\,\varphi, \mathrm{id}, \mathrm{id}) = \overline{\mathcal{S}}_\varphi$ for a surgery datum $\varphi\colon S^{k-1} \times D^{d-k} \hookrightarrow M$ with $d - k \geq 3$.*

Proof. Let $V \coloneqq (V, \psi_0, \psi_1)\colon M_0 \rightsquigarrow M_1$ be a θ-cobordism. By Lemma B.4, there exists a θ-cobordism $V'\colon M_0 \rightsquigarrow M_1$ in the same cobordism class such that (V', M_1) is 2-connected. We define $\mathcal{S}_V \coloneqq \overline{\mathcal{S}}_{V'}$. By definition of $\overline{\mathcal{S}}$ it is clear that this is a functor satisfying the three conditions. Therefore it remains to show that this is well-defined. Let $X\colon V_0 \rightsquigarrow V_1$ be a θ-cobordism relative to $\partial V_0 = \partial V_1$ and let $X_i\colon V_i \rightsquigarrow V_i'$ be relative θ-cobordisms such that (V_i', M_1) is 2-connected for $i = 0, 1$. We get a θ-cobordism $\widetilde{X} \coloneqq X_0^{\mathrm{op}} \cup X \cup X_1\colon V_0' \rightsquigarrow V_0 \rightsquigarrow V_1 \rightsquigarrow V_1'$. Again, by Lemma B.4, we may assume that $(\widetilde{X}, V_i')$ is 2-connected. So, V_1' is obtained from V_0' by a sequence of surgeries of index $k \in \{3, \dots, d-2\}$ by Lemma 1.6.5. One can order these surgeries, so that one first performs the 3-surgeries, the 4-surgeries next and so on up to the $d-3$-surgeries. By Lemma 3.2.1 all of these do not change the homotopy class of $\overline{\mathcal{S}}$ and we may assume that V_1' is obtained from V_0' by a finite sequence of $d-2$-surgeries. Reversing these surgeries we deduce that V_0' is obtained from the admissible cobordism V_1' by a finite sequence of 3-surgeries and by Lemma 3.2.1 the map $\overline{\mathcal{S}}_{V_0'}$ is homotopic to $\overline{\mathcal{S}}_{V_1'}$ and hence $\mathcal{S}$ is well-defined. □

Remark 3.3.2. It follows that $\mathcal{S}\colon \Omega^\theta_d(M_0, M_1) \longrightarrow [\mathcal{R}^+(M_0), \mathcal{R}^+(M_1)]$ is a $\Gamma^\theta(M_1)$-equivariant map with respect to the actions given by disjoint union with the mapping torus on the left and by composition with the pullback map on the right.

4

Applications

In this chapter we give several applications of the main Theorem 3.3.1. We first give a rigidity theorem for the action of the θ-mapping class group on the space of psc metrics. The first application is the obvious one: the computation of examples. In Section 4.1 we present many cases in wich the rigidity theorem applies. The second application is also a quite canonical one: Knowledge about the action map yields knowledge about the quotient $\mathcal{R}^+(M)/\mathrm{Diff}(M)$. In Section 4.2 we explain how certain detection results for $\pi_0(\mathcal{R}^+(M))$ descend to the observer moduli space $\pi_0(\mathcal{M}^+_{x_0}(M))$. We also detect new elements of $\pi_1(\mathcal{M}^+_{x_0}(M))$ for certain manifolds M. The third application is not such an obvious one: Using Theorem 3.3.1 we define H-space structures on $\mathcal{R}^+(M)$ in Section 4.3. We also show that invertible elements with respect to these structures are intrinsic to $\mathcal{R}^+(M)$ and all the obtained structures are in fact equivalent but not equal. As our final application we provide a triviality and a non-triviality criterion for the action map in Section 4.4. This leads to a full characterization of the action of $\mathrm{Diff}^+(M)$ on $\mathcal{R}^+(M)$ for simply connected Spin-manifolds of dimension 7.

4.1 The action of $\Gamma^\theta(M,\hat{l})$ on $\mathcal{R}^+(M)$

For the first application we consider the case $M = M_0 = M_1$. For a space X let $\mathrm{h}\mathbf{Aut}(X)$ denote the group-like H-space of weak homotopy equivalences of X.

Corollary 4.1.1. *Let $d \geq 7$ and let $\theta\colon B \to B\mathrm{O}(d)$ be the stabilized tangential 2-type of M^{d-1}. Then there is a group homomorphism $\mathcal{SE}\colon \Omega_d^\theta \to \pi_0\big(\mathrm{h}\mathbf{Aut}(\mathcal{R}^+(M))\big)$ such that the following diagram commutes:*

$$\begin{array}{ccccc}
\Gamma_\theta(M,\hat{l}) & & \longrightarrow & & \pi_0(\mathrm{h}\mathbf{Aut}(\mathcal{R}^+(M))) \\
[f] & \longmapsto & & (g \mapsto f^*g) & \mathcal{SE}(W) \\
& \searrow & & \nearrow & \\
& [T_f] \;\hookleftarrow & \Omega_d^\theta & [W] \mapsto &
\end{array}$$

$$\Gamma_\theta(M,\hat{l}) \to \Omega_d^\theta,\ [f] \mapsto [T_f], \qquad \Omega_d^\theta \to \pi_0(\mathrm{h}\mathbf{Aut}(\mathcal{R}^+(M))),\ [W] \mapsto \mathcal{SE}(W)$$

Proof. Using the isomorphism $\Phi\colon \Omega_d^\theta \to \Omega_d^\theta(M,M)$ given by disjoint union with $M \times [0,1]$ (cf. Corollary 1.3.7) we define $\mathcal{SE}(W) \coloneqq \mathcal{S}_{(M\times I \amalg W,\mathrm{id},\mathrm{id})}$. Then

$$\begin{aligned}
\mathcal{SE}(W \amalg V) &= \mathcal{S}_{(M\times I \amalg W \amalg V,\mathrm{id},\mathrm{id})} = \mathcal{S}_{((M\times[0,1] \amalg V)\cup(M\times[1,2] \amalg W),\mathrm{id},\mathrm{id})} \\
&= \mathcal{S}_{(M\times[1,2] \amalg V,\mathrm{id},\mathrm{id})} \circ \mathcal{S}_{(M\times[0,1] \amalg W,\mathrm{id},\mathrm{id})} = \mathcal{SE}(W) \circ \mathcal{SE}(V),
\end{aligned}$$

so it is a homomorphism. By Theorem 3.3.1 the above diagram is commutative since $[M \times I \amalg T_\psi, \mathrm{id}, \mathrm{id}] = [M \times I, \mathrm{id}, \psi^{-1}]$ (cf. Corollary 1.3.9 and Remark 1.3.10). □

Remark 4.1.2. As mentioned in Lemma 1.7.12 (see also [Wal11]), Walsh constructed a psc metric G on an admissible self-cobordism $W\colon M \rightsquigarrow M$ extending a given psc metric g_0 on the incoming boundary using basically the same method used here. He showed that the homotopy class of G restricted to the outgoing boundary does not depend on the handle presentation [Wal14, Theorem 1.3]. Therefore he obtained a map $f_W \in \mathrm{Aut}(\pi_0(\mathcal{R}^+(M)))$ given by $[g_0] \mapsto [G|_{M\times\{1\}}]$. By seperating the cobordism part of the picture (Chapter 2) from the scalar curvature part of the picture (Chapter 3) we upgraded this to give an actual homotopy class of a map $\mathcal{S}_W \in \pi_0(\mathrm{h}\mathbf{Aut}(\mathcal{R}^+(M)))$ inducing Walsh's map on $\pi_0(\mathcal{R}^+(M))$.

Having Corollary 4.1.1 at our disposal it is natural to look for cases it applies to. For example, since $\Omega_7^{\mathrm{Spin}} \cong 0 \cong \Omega_7^{SO}$ (cf. [Tho54, Théorème II.16, p. 49] and Proposition 4.1.5), one obtains an immediate result for 6-manifolds:

Corollary 4.1.3. *Let M^6 be a simply connected manifold. Then the action of* $\mathrm{Diff}^+(M)$ *on* $\mathcal{R}^+(M)$ *is homotopy-trivial, i. e. for every orientation preserving diffeomorphism f of M the pullback map f^* is homotopic to the identity.*

4.1.1 Cobordism classes of mapping tori

Having Corollary 4.1.1 at hand we can start the hunt for examples. In this subsection we compute cobordism classes of mapping tori. Except for one, all of the results here have implications to the action of the mapping class group on psc metrics. Let us start by listing a few facts about Ω_*^{SO} and Ω_*^{Spin}. For a manifold M we denote by $p_i(M) \in H^{4i}(M;\mathbb{Z})$ its Pontryagin classes and by $w_i(M) \in H^i(M;\mathbb{Z}/2)$ its Stiefel-Whitney classes[1]. A Pontryagin- or Stiefel-Whitney-number is the integration of a product of Pontryagin- or Stiefel-Whitney-classes against the fundamental class of M.

Proposition 4.1.4 ([Wal60, Corollary 1]). *Let $[T] \in \Omega_d^{\mathrm{SO}}$. If all Pontryagin-numbers of T vanish, then T is rationally nullbordant. If furthermore all Stiefel-Whitney-numbers vanish, then T is nullbordant.*

Lemma 4.1.5 ([Wal60, Theorem 1] and [ABP67, Corollary 2.6]). *Let $\alpha\colon \Omega_d^{\mathrm{Spin}} \to \Omega_d^{\mathrm{SO}}$ denote the forgetful map. We have:*

1. *All torsion in Ω_d^{SO} and Ω_d^{Spin} is 2-torsion.*
2. *$\Omega_d^{\mathrm{SO}} \otimes \mathbb{Q}$ is concentrated in degrees divisible by 4 and $\alpha \otimes \mathrm{id}_{\mathbb{Q}}$ is an isomorphism.*
3. *$\ker \alpha$ is concentrated in degrees $d \equiv 1, 2(8)$ and is a finite dimensional $\mathbb{Z}/2$-vector space there.*

Proposition 4.1.6 ([Neu71]). *The signature of a mapping torus vanishes.*

Proof. The absolute value of the signature of a manifold X^{4n} is bounded by the $2n$-th (rational) Betti number $b_{2n}(X,\mathbb{Q})$. For a mapping torus we have

$$b_{2n}(T_f,\mathbb{Q}) \leq b_{2n}(M,\mathbb{Q}) + b_{2n-1}(M,\mathbb{Q})$$

because of the Wang sequence (see Lemma B.5). So for any f the absolute value of the signature of T_f is bounded by some constant C independent of f. But the mapping torus construction is a homomorphism (see Corollary 1.3.9) and we get:

$$|\mathrm{sign}(T_f)| = \frac{1}{l} \cdot |\mathrm{sign}(T_{f^l})| \leq \frac{C}{l} \overset{l\to\infty}{\longrightarrow} 0. \qquad \square$$

[1] For an introduction to characteristic classes see [Hat17, Chapter 3].

For computations of Stiefel-Whitney-numbers we need to use Steenrod squares. Let us recall their basic properties.

Lemma 4.1.7 ([Bre93, Chapter VI, Section 16]). *For every $i \in \mathbb{N}$, there are natural, additive homomorphisms $\mathrm{Sq}^i\colon H^n(X;\mathbb{Z}/2) \to H^{n+i}(X;\mathbb{Z}/2)$ such that*

1. *For $a \in H^n(X;\mathbb{Z}/2)$, we have $\mathrm{Sq}^0(a) = a$, $\mathrm{Sq}^n(a) = a^2$ and $\mathrm{Sq}^i(a) = 0$ for $i > n$.*
2. $\mathrm{Sq}^n(a \cup b) = \sum_{i+j=n} \mathrm{Sq}^i(a) \cup \mathrm{Sq}^j(b)$ *(Cartan Formula).*
3. $\mathrm{Sq}^i(w_j(\xi)) = \sum_{t=0}^{i} \binom{j+t-i-1}{t} w_{i-t}(\xi) \cup w_{j+t}(\xi)$ *(Wu Formula).*
4. Sq^1 *is the* $\mathrm{mod}\,2$*-reduction of the Bockstein homomorphism for the sequence*

$$\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2.$$

Remark 4.1.8. ad 3. For a proof, see [MT91, pp. 141].

ad 4. In [Bre93] the fourth property is not actually stated but easily deduced. It is said that Sq^1 is the Bockstein homomorphism for the sequence $\mathbb{Z}/2 \xrightarrow{\cdot 2} \mathbb{Z}/4 \longrightarrow \mathbb{Z}/2$. We get a commutative diagram where the rows are exact

$$\begin{array}{ccccccc}
\cdots \longrightarrow & H^k(X;\mathbb{Z}/4) & \longrightarrow & H^k(X;\mathbb{Z}/2) & \xrightarrow{\mathrm{Sq}^1} & H^{k+1}(X;\mathbb{Z}/2) & \to \cdots \\
& \uparrow & & \uparrow \mathrm{id} & & \uparrow \mathrm{mod}\ 2 & \\
\cdots \longrightarrow & H^k(X;\mathbb{Z}) & \longrightarrow & H^k(X;\mathbb{Z}/2) & \xrightarrow{\beta} & H^{k+1}(X;\mathbb{Z}) & \longrightarrow \cdots
\end{array}$$

which implies that $\beta \bmod 2 = \mathrm{Sq}^1$.

Proposition 4.1.9. *The total Stiefel-Whitney class of $\mathbb{CP}^k$ is given by $w(\mathbb{CP}^k) = (1+a)^{k+1}$, where a is the (non-zero) second Stiefel-Whitney class of the tautological complex line bundle. In particular, $w_{2n}(\mathbb{CP}^k) = \binom{k+1}{n} a$.*

Proof. By [MS74, Proof of Theorem 14.10] $T\mathbb{CP}^k \oplus \underline{\mathbb{C}} \cong \overline{\gamma}_1^{k+1}$ where $\overline{\gamma}_1$ is the dual of the tautological bundle over $\mathbb{CP}^k$. We get $w(\mathbb{CP}^k) = w(\overline{\gamma}_1)^{k+1} = (1 + w_2(\gamma_1))^{k+1}$. □

Proposition 4.1.10. *Let M^{d-1} be a closed, oriented manifold. Let $f\colon M \xrightarrow{\cong} M$ be an orientation-preserving diffeomorphism.*

1. *If $d \not\equiv 0(4)$, then $2[T_f] = 0 \in \Omega_d^{\mathrm{SO}}$*
2. *If $d \equiv 0(4)$ and all Pontryagin-classes of M vanish, then $2[T_f] = 0 \in \Omega_d^{\mathrm{SO}}$.*

Proof. By Lemma 4.1.5, it is enough to show that T_f is rationally orientedly nullbordant. The first statement of the Proposition is immediate from Lemma 4.1.5, so let us assume $4|d$. By Proposition 4.1.4 it suffices to show that all Pontryagin-numbers vanish. Consider the Wang sequence (see Lemma B.5):

$$0 \longrightarrow H^n(M)_f \xrightarrow{\delta} H^{n+1}(T_f) \xrightarrow{\iota^*} H^{n+1}(M)^f \longrightarrow 0$$

The righthand map is induced by the inclusion and we have $\iota^* p_i(T_f) = p_i(\iota^* T_f) = p_i(M) = 0$ by our assumption on M. So, all Pontryagin classes of T_f lie in the image of δ. But δ is a boundary map and hence $\delta(x) \cup \delta(y) = 0$ for all $x, y \in H^n(M)$. Therefore, all mixed Pontryagin numbers vanish and the only possibly non-zero one is $\langle p_{d/4}(T_f), [T_f]\rangle$. But this is some nonzero multiple of the signature of T_f which is 0 by Proposition 4.1.6. □

Corollary 4.1.11. *Let M^{d-1} be a closed,* Spin*-manifold. Let $f\colon M \xrightarrow{\cong} M$ be a* Spin*-diffeomorphism.*

1. *If $d \not\equiv 0(4)$, then $2[T_f] = 0 \in \Omega_d^{\mathrm{Spin}}$*
2. *If $d \equiv 0(4)$ and all Pontryagin-classes of M vanish, then $2[T_f] = 0 \in \Omega_d^{\mathrm{Spin}}$.*

Proof. This is immediate from Proposition 4.1.5 and Proposition 4.1.10. □

Corollary 4.1.12. *Let M^{d-1} be a closed, oriented manifold with finite fundamental group G. Let $a\colon M \to BG$ be the classifying map for the universal cover of M and let $f\colon M \xrightarrow{\cong} M$ be an orientation preserving diffeomorphism that acts by an inner automorphism on fundamental group. We get a map $a_f\colon T_f \to BG$. If $4|d$ let $p_i(M) = 0$ for all $i \geq 0$. Then there exists an $n \in \mathbb{N}$ such that $0 = n \cdot [T_f, a_f] \in \Omega_d^{\mathrm{SO}}(BG)$. Furthermore n divides $2 \cdot |G|$.*

Proof. Consider the Atiyah-Hirzebruch-spectral-sequence with rational coefficients:

$$E^2_{pq} = H_p(BG, \Omega_q^{\mathrm{SO}} \otimes \mathbb{Q}) \Rightarrow \Omega_{p+q}^{\mathrm{SO}}(BG) \otimes \mathbb{Q}$$

Since $H_p(BG)$ is torsion for $p \geq 1$, $E^2_{pq} = 0$ unless $p = 0$. If $p = 0$, then $E_2^{pq} \cong \Omega_d^{SO} \otimes \mathbb{Q}$ and hence by convergence of the spectral sequence $\Omega_d^{\mathrm{SO}}(BG) \otimes \mathbb{Q} \cong \Omega_d^{\mathrm{SO}} \otimes \mathbb{Q}$. The same proof as in Proposition 4.1.10 applies. For the divisibility, note that for any element $x \in H_p(BG)$ satisfies $|G| \cdot x = 0$. □

Remark 4.1.13. Again, the analogous result is true if one replaces SO by Spin.

Putting further restrictions on M we get the following result, which is due to Kreck [Kre76, Proposition 13]. For the sake of completeness we include the proof here.

Proposition 4.1.14. *Let M be stably parallelizable and $f\colon M \xrightarrow{\cong} M$ be an orientation preserving diffeomorphism. Then T_f is orientedly nullbordant. If furthermore f is a Spin diffeomorphism and $d \not\equiv 1,2(8)$, then T_f is Spin nullbordant.*

Proof. By Proposition 4.1.4 it suffices to show that all characteristic numbers vanish. By the same argument as in the proof of Proposition 4.1.10, all mixed Pontryagin and Stiefel-Whitney numbers vanish. It remains to consider $\langle p_{d/4}(T_f), [T_f]\rangle$ and $\langle w_d(T_f), [T_f]_{\mathbb{Z}/2}\rangle$. The former vanishes by the same argument as in the proof of Proposition 4.1.10. The latter is the $\mathrm{mod}2$-reduction of the Euler number. But the Euler number of a fibration is multiplicative and hence $\langle w_d(T_f), [T_f]_{\mathbb{Z}/2}\rangle = 0$. The Spin-case follows directly from Lemma 4.1.5. □

Proposition 4.1.15. *Let $k \geq 1$ and let $f\colon \mathbb{CP}^{2k+1} \xrightarrow{\cong} \mathbb{CP}^{2k+1}$ be a Spin-diffeomorphism. Then T_f is Spin-nullbordant.*

Proof. A mapping torus of $\mathbb{CP}^{2k+1}$ has real dimension $4k+3$ and hence all Pontryagin numbers vanish for dimension reasons. So it suffices to consider Stiefel-Whitney numbers. Since f is orientation preserving, it must act trivially on the highest cohomology of $\mathbb{CP}^{2k+1}$. It follows from the ring structure that f acts trivially on the entire cohomology ring as $2k+1$ is odd. From Lemma B.5 we get the following decomposition

$$H^n(T_f) \cong H^n(\mathbb{CP}^{2k+1}) \oplus H^{n-1}(\mathbb{CP}^{2k+1})$$

Therefore $\iota^*\colon H^{2l}(T_f) \to H^{2l}(\mathbb{CP}^{2k+1})$ is an isomorphism and all odd Stiefel-Whitney classes of T_f lie in the image of the boundary map δ. This implies that mixed Stiefel-Whitney numbers may contain at most one odd Stiefel-Whitney class, as the boundary map kills products. Now $w_{2n}(\mathbb{CP}^{2k+1}) = \binom{2k+2}{n}a$ which is $0 \bmod 2$ if n is odd. Furthermore, by the Wu formula we have

$$\begin{aligned} w_{4l+3}(T_f) &= w_1(T_f) \cup w_{4l+2}(T_f) + w_{4l+3}(T_f) = \mathrm{Sq}^1(w_{2(2l+1)}(T_f)) \\ &= (\iota^*)^{-1}\mathrm{Sq}^1(w_{2(2l+1)}(\mathbb{CP}^{2k+1})) = 0 \end{aligned}$$

since T_f is orientable. So, any possibly nonzero Stiefel-Whitney number has the form

$$\langle w_{4n_1}(T_f)\cdots w_{4n_{l-1}}(T_f), [T_f]_{\mathbb{Z}/2}\rangle \text{ or}$$
$$\langle w_{4n_1}(T_f)\cdots w_{4n_{l-1}}(T_f)\cdot w_{4n_l+1}(T_f), [T_f]_{\mathbb{Z}/2}\rangle.$$

However, the degree of $w_{4n_1}(T_f)\cdots w_{4n_{l-1}}(T_f)\cdot w_{4n_l+1}(T_f)$ is $\equiv 0, 1(4)$ but the dimension of T_f is $\equiv 3(4)$. So, all Stiefel-Whitney numbers vanish and T_f is orientedly and even Spin-nullbordant by Lemma 4.1.5. □

An example of a mapping torus which is not nullbordant is the following.

Proposition 4.1.16. *The mapping torus T_c of the diffeomorphism $c\colon \mathbb{CP}^{2k} \xrightarrow{\cong} \mathbb{CP}^{2k}$ induced by complex conjugation which is orientation-preserving is not orientedly nullbordant.*

Remark 4.1.17. The complex conjugation on $\mathbb{CP}^1 \cong S^2$ is homotopic to the antipodal map and hence induces -1 on $H^2(\mathbb{CP}^1) \hookrightarrow H^2(\mathbb{CP}^{2k})$. Because of the ring structure it is orientation preserving on $\mathbb{CP}^{2k}$ but not on $\mathbb{CP}^{2k+1}$.

Proof of Proposition 4.1.16. This proof is a generalization of a math-overflow post by Achim Krause [Kra]. For this entire proof, w_i denotes the i-th Stiefel-Whitney class of T_c. We have $c^* = (-1)^n$ on $H^{2n}(\mathbb{CP}^{2k})$ and hence we get from Lemma B.5

$$H^{2n}(T_c) \cong H^{2n}(\mathbb{CP}^{2k})^{c^*} \cong \begin{cases} \mathbb{Z} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$
$$H^{2n+1}(T_c) \cong H^{2n}(\mathbb{CP}^{2k})_{c^*} \cong \begin{cases} \mathbb{Z} & \text{if } n \text{ is even} \\ \mathbb{Z}/2 & \text{if } n \text{ is odd} \end{cases}$$

and $H^n(T_c;\mathbb{Z}/2) = \mathbb{Z}/2$ for $0 \leq n \leq 4k+1$. We get the Bockstein sequence

$$\begin{array}{ccccccc} \cdots \longrightarrow H^{4n+2}(T_c;\mathbb{Z}) & \longrightarrow & H^{4n+2}(T_c;\mathbb{Z}/2) & \xrightarrow{\beta} & H^{4n+3}(T_c;\mathbb{Z}) & \longrightarrow & H^{4n+3}(T_c;\mathbb{Z}) = \mathbb{Z}/2 \\ \| & & \| & & \| & & \downarrow \\ 0 & & \mathbb{Z}/2 & & \mathbb{Z}/2 & & H^{4n+3}(T_c;\mathbb{Z}/2) = \mathbb{Z}/2 \\ & & & & & & \downarrow \\ & & & & & & \mathbb{Z} = H^{4n+4}(T_c;\mathbb{Z}) \longrightarrow \cdots \end{array}$$

If $n \not\equiv 2(4)$, the map $\beta = 0$ because in these cases $H^{n+1}(T_c, \mathbb{Z})$ is either 0 or $\mathbb{Z}$. So the integral Bockstein homomorphism $\beta\colon H^n(T_c, \mathbb{Z}/2) \to H^{n+1}(T_c, \mathbb{Z})$ is nontrivial if and only if $n \equiv 2(4)$. The same holds for Sq^1 and we have $0 \neq \mathrm{Sq}^1(w_2) = w_3$ by Wu's formula. By Cartan's formula

$$\mathrm{Sq}^1(w_2^n) = \mathrm{Sq}^1(w_2) \cup w_2^{n-1} + w_2 \cup \mathrm{Sq}^1(w_2^{n-1})$$

which by induction is 0 if n is even and equal to $w_2^{n-1} \cup w_3$ if n is odd. This is nonzero because Sq^1 is nonzero. We compute further

$$\begin{aligned}\mathrm{Sq}^2(w_2^{n-1} \cup w_3) = {} & \mathrm{Sq}^2(w_2) \cup w_2^{n-2} \cup w_3 \\ & + w_2 \cup \mathrm{Sq}^2(w_2^{n-2} \cup w_3).\end{aligned}$$

By induction this is 0 if n is even and equal to $w_2^n \cup w_3$ if n is odd. Also, if $w_2^{n-1} \cup w_3$ is nonzero, it must lie in the image of δ for degree reasons and we get for n odd

$$\mathrm{Sq}^2(w_2^{n-1} \cup w_3) = \mathrm{Sq}^2(\delta(w_2(\mathbb{CP}^{2k})^n)) = \delta \mathrm{Sq}^2(w_2(\mathbb{CP}^{2k})^n)$$

which is nonzero by the same computation as for Sq^1. So, for $n = 2k-1$ we get

$$\mathrm{Sq}^2(\mathrm{Sq}^1(w_2^{2k-1})) = \underbrace{\mathrm{Sq}^2(w_2^{2k-2} \cup w_3)}_{\neq 0} = w_2^{2k-1} \cup w_3$$

and we found a non-vanishing Stiefel-Whitney number. □

Remark 4.1.18. If k is odd, $\langle w_2(T_f) \cup w_{4k-1}(T_f), [T_f] \rangle$ is another nonvanishing Stiefel-Whitney number.

Proposition 4.1.19. *Let X^{2k}, $k \geq 3$ be a stably parallelizable, simply connected, closed manifold and let $H^{2k-i}(X; \mathbb{Z}/2) = 0$ for $i = 3, 5$. Let $f\colon X \times \mathbb{CP}^2 \xrightarrow{\cong} X \times \mathbb{CP}^2$ be an orientation preserving diffeomorphism. Then T_f is orientedly nullbordant.*

Proof. Again, we only need to compute Stiefel-Whitney numbers of T_f, which we again write as w_i. All cohomology in this proof will be with $\mathbb{Z}/2$ coefficients. First, we note that $w_2(\mathbb{CP}^2) \neq 0$ and $w_4(\mathbb{CP}^2) = w_2(\mathbb{CP}^2)^2 \neq 0$. Also, all Stiefel-Whitney classes of X vanish. So, all w_i except for w_2 and w_4 are in the image of δ. We further have $H^2(T_f) \cong H^1(X \times \mathbb{CP}^2)_{f^*} \oplus H^2(X \times \mathbb{CP}^2)^{f^*} \cong H^2(X \times \mathbb{CP}^2)^{f^*}$ because X is simply connected. Hence we get $w_2(T_f) = (\iota^*)^{-1}(w_2(X \times \mathbb{CP}^2)) = (\iota^*)^{-1}(w_2(\mathbb{CP}^2))$. Again, products in the image of δ are 0 and so the only possibly nonzero Stiefel-Whitney numbers correspond to the classes $w_{2k+5}, w_{2k+3}w_2$ and $w_{2k+1}w_2^2$, since $w_2^3 = (\iota^*)^{-1}w_2(\mathbb{CP}^2)^3 = 0$. By Wu's formula we have

$$\mathrm{Sq}^1(w_{2k+i-1}) = w_1 \cup w_{2k+i-1} + \binom{2k+i-2}{1} w_{2k+i} = w_{2k+i}$$

for $i = 1, 3, 5$. If $2k \geq 5$, $w_{2k+i-1} \in \operatorname{im} \delta$ and we have

$$\begin{aligned} w_{2k+4} &\in \delta\big(H^{2k+3}(X \times \mathbb{CP}^2)\big) \\ &\cong \delta\big(H^{2k-1}(X) \otimes H^4(\mathbb{CP}^2)\big) \\ w_{2k+2} &\in \delta\big(H^{2k+1}(X \times \mathbb{CP}^2)\big) \\ &\cong \delta\Big(\big(H^{2k-1}(X) \otimes H^2(\mathbb{CP}^2)\big) \oplus \big(H^{2k-3}(X) \otimes H^4(\mathbb{CP}^2)\big)\Big) \\ w_{2k} &\in \delta\big(H^{2k-1}(X \times \mathbb{CP}^2)\big) \\ &\cong \delta\Big(\big(H^{2k-1}(X) \otimes H^0(\mathbb{CP}^2)\big) \oplus \big(H^{2k-3}(X) \otimes H^2(\mathbb{CP}^2)\big) \\ &\quad \oplus \big(H^{2k-5}(X) \otimes H^4(\mathbb{CP}^2)\big)\Big) \end{aligned}$$

By Poincaré duality, Hurewicz' theorem and the universal coefficient theorem it follows that $H^{2k-1}(X) \cong \pi_1(X) \otimes \mathbb{Z}/2 \cong 0$ and so all the groups on the righthand side are 0. □

The proof for the following Proposition is adapted from [KL05, Chapter 16], where they verify the Novikov conjecture for $\mathbb{Z}^n$. They do so by reducing the conjecture to the problem of showing that higher signatures of certain mapping tori vanish. Let N be a manifold and let $a\colon N \to BG$ be a map. The higher signature of (N, a) with respect to some cohomology class $x \in H^*(BG)$ is defined as

$$\operatorname{sign}_x(N, a) := \langle \mathcal{L}(N) \cup a^*x, [N] \rangle$$

where $\mathcal{L}(N)$ is the Hirzebruch $\mathcal{L}$-class of N.

Proposition 4.1.20. *Let X^{d-k-1} be a manifold with vanishing Pontryagin classes such that the Whitehead group $\mathrm{Wh}(\pi_1 X \oplus \mathbb{Z}^m)$ is trivial for $m \in \{0, \dots, k-1\}$[2]. Let $M := X \times \mathbb{T}^k$ and let $f\colon M \xrightarrow{\cong} M$ be an orientation preserving diffeomorphism that acts on $\pi_1(\mathbb{T}^k) \subset \pi_1(M)$ by an inner automorphism. We then get a map $a_f\colon T_f \to B\mathbb{Z}^k$ such that $M \to T_f \xrightarrow{a_f} B\mathbb{Z}^k$ is homotopic to the projection map. Then $0 = n \cdot [T_f, a_f] \in \Omega_d^{\mathrm{SO}}(B\mathbb{Z}^k)$ for some $n \in \mathbb{N}$.*

[2]This is fulfilled for example if X is simply connected or $\pi_1(X) = \mathbb{Z}^n$ (cf. [BHS64, p.63])

Proof. Recall that for any manifold N, the class of $[N,a]$ vanishes in $\Omega_d^{\mathrm{SO}}(X)\otimes\mathbb{Q}$, if for all classes $x\in H^n(X;\mathbb{Q})$, the characteristic number $\langle p_I(N)\cup a^*x,[N]\rangle$ vanishes for every multi index $I=(i_1,\dots,i_{d/4})$. So let us now take the situation as described in the Proposition. By the same argument as in the proof of Proposition 4.1.10, all products of Pontryagin classes of T_f are zero and hence we only need to check characteristic numbers of the form $\langle p_{(d-n)/4}(T_f)\cup a_f^*x,[T_f]\rangle$. But in our situation, this is precisely a multiple of the higher signature of (T_f,a_f) associated to x and hence it suffices to show that all higher signatures of mapping tori vanish.

Let $x\in H^m(B\mathbb{Z}^k)$. Then there exists a projection map $p\colon B\mathbb{Z}^k\twoheadrightarrow B\mathbb{Z}^m$ such that $x=c\cdot p^*u_m$ for some $c\in\mathbb{Q}$ and u_m a generator of $H^m(B\mathbb{Z}^m)$. We may assume that the composition $\mathbb{T}^m\hookrightarrow M\hookrightarrow T_f\xrightarrow{a_f}B\mathbb{Z}^k\xrightarrow{p}B\mathbb{Z}^m$ is equal to the projection map since a_f can be changed by a homotopy.

We first show that f is isotopic to a diffeomorphism that fixes $\{x\}\times\mathbb{T}^{k-1}\times X$ setwise for some $x\in S^1$ after possibly passing to a finite covering[3]. We abbreviate $X':=\mathbb{T}^{k-1}\times X$. Without loss of generality we may assume that f fixes a point x_0. We pass to the cover $\mathbb{R}\times X'$ and we consider the lifted diffeomorphism $\hat{f}\colon\mathbb{R}\times X'\to\mathbb{R}\times X'$. Now, $\{0\}\times X'$ is compact and there exists an integer $l\geq 0$ such that $\hat{f}(\{0\}\times X')\subset[-l,l]\times X'$. We divide $[-l,l]\times X'=A_-\cup A_+$ where $A_-\cap A_+=\hat{f}(\{0\}\times X')$ and $\partial A_\pm=\{\pm l\}\times X'\amalg\hat{f}(\{0\}\times X')$. The inclusion $A_+\hookrightarrow\hat{f}([0,\infty)\times X')$ is a homotopy equivalence and so A_+ is an h-cobordism. Since the Whitehead group $\mathrm{Wh}(\pi_1X')$ is trivial, there is a diffeomorphism $\rho\colon\hat{f}(\{0\}\times X')\times[0,1]\xrightarrow{\cong}A_+$ with $\rho|_{\hat{f}(\{0\}\times X')\times\{0\}}=\mathrm{id}$ by the s-cobordism theorem. This gives an isotopy $\rho_t\colon\mathbb{R}\times X'\to\mathbb{R}\times X'$ defined by $\rho_t(x):=\rho(\hat{f}(0,x),t)$ with $\rho_0(x)=(0,\hat{f}(0,x))$ and $\operatorname{im}\rho_1=M\times\{l\}$. We now project down to $\mathbb{R}/2l\mathbb{Z}$ and we get a diffeomorphism $\hat{f}_l$ which is the $2l$-fold cover of f. The above isotopy induces an isotopy from $\hat{f}|_{\{0\}\times X'}$ to a diffeomorphism $\{0\}\times X'\xrightarrow{\cong}\{l\}\times X'$ which we can further isotope to a diffeomorphism of $\{0\}\times X'$. By the isotopy extension theorem we get an isotopy of $\hat{f}$ to a diffeomorphism $\hat{f}'$ of $(\mathbb{R}/2l\mathbb{Z})\times X'$ fixing $\{0\}\times X'$ setwise.

By an inductive argument we may assume that f fixes $\{0\}\times\mathbb{T}^{k-m}\times X$. The higher signature of (T_f,a_f) associated to x is the (ordinary) signature of a preimage of a regular value of $p\circ a_f$ ([KL05, Proposition 4.3]). Now 0 is a regular value with $(p\circ a_f)^{-1}(0)=T_{f|_{\{0\}\times\mathbb{T}^{k-m}\times X}}$ which is a mapping torus and has trivial signature by Proposition 4.1.6. □

[3]Passing to a finite cover does not alter the (non-)triviality of the signature because the signature is multiplicative under finite coverings.

4.1.2 Examples of manifolds with trivial actions

In this subsection we are mostly gathering results which are directly implied by the corresponding results from the previous subsection and Corollary 4.1.1. We also note that for simply connected Spin-manifolds the map $\Gamma^{\mathrm{Spin}}(M) \to \pi_0(\mathrm{Diff}^+(M))$ is surjective by Proposition 1.2.8. Hence, we are able to deduce results about the action of the group of orientation preserving diffeomorphisms $\mathrm{Diff}^+(M)$ on $\mathcal{R}^+(M)$ in this case.

Corollary 4.1.21. *Let $d \geq 7$ and let M^{d-1} be a simply connected, closed, oriented manifold. If $d \equiv 0(4)$, let all Pontryagin classes of M vanish. Let $f\colon M \xrightarrow{\cong} M$ be an orientation preserving diffeomorphism. Then $(f^*)^2\colon \mathcal{R}^+(M) \to \mathcal{R}^+(M)$ is homotopic to the identity.*

Corollary 4.1.22. *Let M be a connected, closed,* Spin*-manifold with finite fundamental group and let $f\colon M \xrightarrow{\cong} M$ be a* Spin*-diffeomorphism that acts on $\pi_1(M)$ by an inner automorphism. Then $(f^*)^n\colon \mathcal{R}^+(M) \to \mathcal{R}^+(M)$ is homotopic to the identity for some $n \in \mathbb{N}$. Furthermore, n can be chosen to divide $2|\pi_1(M)|$.*

Corollary 4.1.23. *Let $d \geq 7$ and $d \not\equiv 1, 2(8)$. Let M^{d-1} be a simply connected, stably parallelizable manifold. Then the action of* $\mathrm{Diff}^+(M)$ *on $\mathcal{R}^+(M)$ is homotopy-trivial.*

For $M = S^{d-1}$ we get:

Corollary 4.1.24. *The action of* $\mathrm{Diff}^+(S^{d-1})$ *on $\mathcal{R}^+(S^{d-1})$ in the homotopy category factors through a free $\mathbb{Z}/2$-action if $d \equiv 1, 2(8)$ and is trivial otherwise.*

Proof. The non-triviality is a result by Hitchin [Hit74, Theorem 4.7]. □

This recovers a version of a result of Hajduk:

Proposition 4.1.25 ([Haj88, Theorem 3.6])**.** *The action of* $\mathrm{Diff}^+(S^{d-1})$ *on concordance classes of psc metrics factors through a free $\mathbb{Z}/2$-action if $d \equiv 1, 2(8)$ and is trivial otherwise.*

Corollary 4.1.26. *For $k \geq 1$,* $\mathrm{Diff}^+(\mathbb{CP}^{2k+1})$ *acts homotopy-trivially on $\mathcal{R}^+(\mathbb{CP}^{2k+1})$.*

Corollary 4.1.27. *Let X^{2k}, $k \geq 3$ be a stably parallelizable, simply connected, closed manifold with $H^{2k-i}(X; \mathbb{Z}/2) = 0$ for $i = 3, 5$. Then* $\mathrm{Diff}^+(X \times \mathbb{CP}^2)$ *acts homotopy-trivial on $\mathcal{R}^+(X \times \mathbb{CP}^2)$.*

Remark 4.1.28. Note that $\mathcal{R}^+(\mathbb{CP}^2)$ is nonempty and hence so is $\mathcal{R}^+(\mathbb{CP}^2 \times X)$ for *any* manifold X.

For the final example we note the following: From the Atiyah-Hirzebruch spectral sequence and Proposition 4.1.5 we deduce that $\Omega_*^{\mathrm{Spin}}(X) \otimes \mathbb{Q} \longrightarrow \Omega_*^{\mathrm{SO}}(X) \otimes \mathbb{Q}$ is an isomorphism.

Corollary 4.1.29. *Let X^{d-k-1} be a simply connected,* Spin*-manifold with vanishing Pontryagin classes and let $M := X \times \mathbb{T}^k$. Let $f\colon M \xrightarrow{\cong} M$ be a* Spin*-diffeomorphism that acts on $\pi_1(M)$ by an inner automorphism. Then there exists an $n \in \mathbb{N}$ such that $(f^*)^n\colon \mathcal{R}^+(M) \to \mathcal{R}^+(M)$ is homotopic to the identity.*

4.2 The observer moduli space

One might be tempted to think that knowledge of the homotopy class of the action map $\mathrm{Diff}(M) \to \mathrm{h}\mathbf{Aut}(\mathcal{R}^+(M))$ leads to knowledge of the quotient $\mathcal{R}^+(M)/\mathrm{Diff}(M)$. The problem however is, that the action is not free, as there might exist a metric with nontrivial isometry group. One can fix this in two ways: One replaces the quotient by the Borel construction, i. e. by the *homotopy quotient*, or one restricts to a subgroup of $\mathrm{Diff}(M)$ that acts freely on $\mathcal{R}^+(M)$ and $\mathcal{R}(M)$. Under some assumptions this subgroup will automatically consist of orientation preserving or even Spin-diffeomorphisms which makes the results from the previous section applicable. So we pursue the latter idea which originates from [AB02]. Let us start by giving definitions.

Definition 4.2.1. For $x_0 \in M$ we define $\mathrm{Diff}_{x_0}(M)$ to be the subgroup of $\mathrm{Diff}(M)$ consisting of all diffeomorphisms f satisfying $f(x_0) = x_0$ and $\mathrm{d}f_{x_0} = \mathrm{id}\colon T_{x_0}M \to T_{x_0}M$.

The following Lemma is a standard exercise in differential geometry. A proof can be found in [BHSW10, Lemma 1.2] or in [TW15, Lemma 7.1.2].

Lemma 4.2.2. *Let M be connected. Then the action of $\mathrm{Diff}_{x_0}(M)$ on $\mathcal{R}(M)$ and hence on $\mathcal{R}^+(M)$ is free.*

Definition 4.2.3. We define the *observer moduli spaces* by $\mathcal{M}_{x_0}(M) := \mathcal{R}(M)/\mathrm{Diff}_{x_0}(M)$ and $\mathcal{M}_{x_0}^+(M) := \mathcal{R}^+(M)/\mathrm{Diff}_{x_0}(M)$.

Since the action of $\mathrm{Diff}_{x_0}(M)$ is free we get a fiber bundle

$$\mathrm{Diff}_{x_0}(M) \to \mathcal{R}^+(M) \to \mathcal{M}_{x_0}^+(M)$$

and hence a long exact sequence of homotopy groups ending in

$$\cdots \to \pi_1(\mathcal{M}^+_{x_0}(M)) \to \pi_0(\mathrm{Diff}_{x_0}(M)) \to \pi_0(\mathcal{R}^+(M)) \to \pi_0(\mathcal{M}^+_{x_0}(M)) \to *$$

Now, we need to relate $\mathrm{Diff}_{x_0}(M)$ to $\mathrm{Diff}^\theta(M)$. Since for any $f \in \mathrm{Diff}_{x_0}(M)$, $\mathrm{d}f_{x_0}$ is the identity, we can isotope f to be the identity on a small neighbourhood U of x_0. We get an isomorphism $\mathrm{Diff}_{x_0}(M) \xrightarrow{\cong} \mathrm{Diff}_\partial(M \setminus U)$ to the group of diffeomorphisms restricting to the identity on a neighbourhood of the boundary. We also have $\pi_0(\mathrm{Diff}_\partial(M \setminus U)) \cong \pi_1(B\mathrm{Diff}_\partial(M \setminus U))$. Let $\hat{l}_\partial$ be a θ-structure on $\partial(M \setminus U)$. Analogously to Definition 1.2.4 we define for a fibration:

$$B\mathrm{Diff}^\theta_\partial(M \setminus U) := E\mathrm{Diff}_\partial(M \setminus U) \underset{\mathrm{Diff}_\partial(M\setminus U)}{\times} \mathrm{Bun}_\partial(T(M \setminus U) \oplus \mathbb{R}, \theta^* U_d)$$

where Bun_∂ denotes the space of bundle maps that are equal to $\hat{l}_\partial$ on the boundary. We have the following lemma which is a special case of [GRW14, Lemma 7.16].

Lemma 4.2.4. $\mathrm{Bun}_\partial(T(M \setminus U), \theta^* U_d) \simeq \mathrm{pt}$ *in either of two following cases:*

1. *M is a 2-connected Spin-manifold and $B = B\mathrm{Spin}(d)$.*
2. *M is a simply connected orientable manifold and $B = B\mathrm{SO}(d)$.*

Proof. Let us abbreviate $N := M \setminus U$. The inclusion $\partial N \hookrightarrow N$ is 2-connected in the first and 1-connected in the second case. Therefore by [GRW14, Lemma 7.16] in both cases $\mathrm{Bun}_\partial(TN, \theta^* U_d) \simeq \mathrm{Bun}_\partial(TN, U_d) \simeq \mathrm{pt}$. □

By Corollary 4.1.1 we get a commutative diagram in both of these cases:

$$\begin{array}{ccccccc}
\cdots \to \pi_0(\mathrm{Diff}_{x_0}(M)) & \longrightarrow & \pi_0(\mathcal{R}^+(M)) & \longrightarrow & \pi_0(\mathcal{M}^+_{x_0}(M)) & \longrightarrow & 0 \\
\cong \downarrow & & \uparrow & & & & \\
\pi_1(B\mathrm{Diff}_{x_0}(M)) & & & & & & \\
\cong \downarrow & & & & & & \\
\pi_1(B\mathrm{Diff}_\partial(M \setminus U)) & & & & & & \\
\cong \downarrow & & & & & & \\
\pi_1(B\mathrm{Diff}^\theta_\partial(M \setminus U)) & & & & & & \\
\downarrow & & & & & & \\
\pi_1(B\mathrm{Diff}^\theta(M)) & \longrightarrow & \Omega^\theta_d & & & & \\
[f] & \longmapsto & [T_f] & & & &
\end{array}$$

So, as soon as the map $\Gamma^\theta(M) \longrightarrow \Omega^\theta_d$ is trivial we get a bijection $\pi_0(\mathcal{R}^+(M)) \xrightarrow{\cong} \pi_0(\mathcal{M}^+_{x_0}(M))$ and a surjection $\pi_1(\mathcal{M}^+_{x_0}(M)) \twoheadrightarrow \pi_1(B\mathrm{Diff}_\partial(M \setminus U))$. This happens for example when $d = 7$, because $\Omega^\theta_7 = 0$ in both cases.

We now give two detection results for the observer moduli space. The first is obtained by applying the above to the work of Botvinnik, Ebert and Randal-Williams [BERW17] together with Corollary 4.1.21:

Theorem 4.2.5. *Let $d \geq 7$ and let M^{d-1} be a 2-connected* Spin*-manifold.*

1. *If $d \equiv 0(4)$ and all Pontryagin classes of M vanish, the space $\mathcal{M}^+_{x_0}(M)$ has infinitely many path components.*
2. *If $d \not\equiv 1, 2(8)$ and M is stably parallelizable, the map $\pi_0(\mathcal{R}^+(M)) \longrightarrow \pi_0(\mathcal{M}^+_{x_0}(M))$ is a bijection.*

At the other end of the sequence we get a detection result for $\pi_1(\mathcal{M}^+_{x_0}(M))$ for special manifolds M using the work of Galatius and Randal-Williams [GRW16] who computed the fundamental group of $B\mathrm{Diff}_\partial(W^{2n}_g)$ for $W^{2n}_g := (S^n \times S^n)^{\#g}$ the high-dimensional genus g surface. Let $BO(d)\langle l\rangle$ denote the l-connected cover of $BO(d)$.

Theorem 4.2.6. *For $g \geq 5$, $n \geq 3$ and $n \not\equiv 0(4)$ there is a surjective map $\pi_1(\mathcal{M}^+_{x_0}(W_g^{2n})) \to \Omega^{\langle n \rangle}_{2n+1} \oplus G_n$ where $\Omega^{\langle n \rangle}_{2n+1}$ denotes the $B\mathrm{O}(2n+1)\langle n \rangle$-cobordism group and*

$$G_n \cong \begin{cases} (\mathbb{Z}/2)^2 & \text{if } n \text{ is even} \\ 0 & \text{if } n = 3, 7 \\ \mathbb{Z}/4 & \text{otherwise.} \end{cases}$$

Proof. First we note that W_g^{2n} is 2-connected, stably parallelizable and Spin and so by Proposition 4.1.14 we get that any mapping torus is Spin-nullbordant (here we use $n \not\equiv 0 \bmod 4$) implying that the map $\pi_1(\mathcal{M}^+_{x_0}(W_g^{2n})) \to \pi_0(\mathrm{Diff}_{x_0}(W_g^{2n}))$ is surjective. Above we computed that

$$\pi_0(\mathrm{Diff}_{x_0}(M)) \cong \pi_1(B\mathrm{Diff}_\partial(W_g^{2n} \setminus D)$$

and by [GRW16, Theorem 1.3] that $\pi_1(B\mathrm{Diff}_\partial(W_g^{2n} \setminus D)$ maps surjectively to $\Omega^{\langle n \rangle}_{2n+1} \oplus G_n$. □

4.3 An H-space structure on $\mathcal{R}^+(M)$

In this section we apply Theorem 3.3.1 to construct a family of H-space multiplications on $\mathcal{R}^+(M)$ for certain manifolds M. We will show that all of these are equivalent and that invertible elements do not depend on the chosen multiplication.

4.3.1 Definition and easy computations

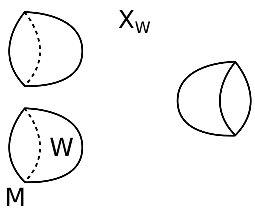


FIGURE 4.1: The θ-cobordism $X_W\colon M \amalg M \rightsquigarrow M$.

Let us first fix the situation for this section: Let $d \geq 7$, let M^{d-1} be a manifold and let $\theta\colon B \to B\mathrm{O}(d)$ be the stabilized tangential 2-type of M. Let $W\colon \emptyset \rightsquigarrow M$ be a θ-nullbordism of M.

This gives a map $\mathcal{S}_W\colon \mathcal{R}^+(\emptyset) \to \mathcal{R}^+(M)$. Note that $\mathcal{R}^+(\emptyset) = \{\mathrm{pt}\}$ and let $e_W := \mathcal{S}_W(\mathrm{pt})$. We also have a θ-cobordism

$$X_W := W^{\mathrm{op}} \amalg W^{\mathrm{op}} \amalg W\colon M \amalg M \rightsquigarrow M$$

as in Figure 4.1. Now, $X_W \in \Omega_d^\theta\big((M \amalg M), M\big)$ and by Theorem 3.3.1, we get a map

$$\mu_W := \mathcal{S}_{X_W} : \mathcal{R}^+(M) \times \mathcal{R}^+(M) \longrightarrow \mathcal{R}^+(M)$$

whose homotopy class only depends on the θ-cobordism class of W.

Theorem 4.3.1. *The map μ_W defines a commutative and associative H-space structure on $\mathcal{R}^+(M)$ and the neutral element is given by e_W.*

Proof. First we show that e_W really is the neutral element. We need to show that the composition

$$\mathcal{R}^+(M) \xrightarrow{\text{id} \times e_W} \mathcal{R}^+(M) \times \mathcal{R}^+(M) \xrightarrow{\mu_W} \mathcal{R}^+(M)$$

is homotopic to the identity. The first map is equal to $\mathcal{S}_{(M\times I)\amalg W}$ and the composition is given by

$$\mathcal{S}_{X_W} \circ \mathcal{S}_{(M\times I)\amalg W} = \mathcal{S}_{(M\times I)\cup W^{\text{op}}\amalg W \amalg dW} \sim \mathcal{S}_{(M\times I)\cup(M\times I)} \sim \text{id}$$

as the double of W is nullbordant by Proposition 1.3.3 (see Figure 4.2).

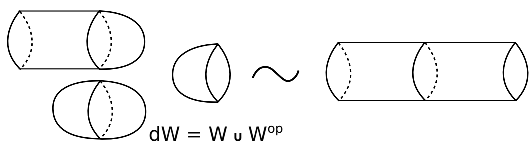


FIGURE 4.2: e_W is the neutral element.

For commutativity, the composition $\mu_W \circ \tau$, where τ is the map switching the factors, has to be homotopic to μ_W. The map τ however is given by the surgery map $\mathcal{S}$ for the cobordism in Figure 4.3 and the composition of this cobordism with X_W is bordant to X_W relative to the boundary (see Figure 4.3). This also implies that e_W is a two-sided unit.

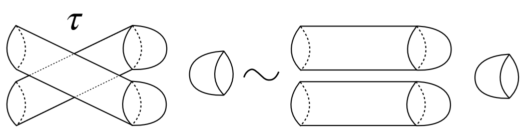


FIGURE 4.3: μ_V is commutative.

For associativity we need to show that the following diagram commutes:

$$\begin{array}{ccccc} & \overset{\mathrm{id}\times\mu_W}{\swarrow} & \mathcal{R}^+(M)\times\mathcal{R}^+(M)\times\mathcal{R}^+(M) & \overset{\mu_W\times\mathrm{id}}{\searrow} & \\ \mathcal{R}^+(M)\times\mathcal{R}^+(M) & & & & \mathcal{R}^+(M)\times\mathcal{R}^+(M) \\ & \underset{\mu_W}{\searrow} & \mathcal{R}^+(M) & \underset{\mu_W}{\swarrow} & \end{array}$$

Again, all maps are given by surgery maps and the proof is finished by Figure 4.4. □

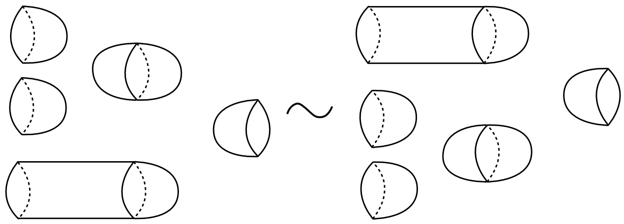


FIGURE 4.4: μ_V is associative.

Remark 4.3.2. 1. This proof shows how easy it is to work with this kind of "graphical calculus". It is always possible to write down the formulas, however the pictorial computation is much more enlightening.

2. A word of warning is appropriate here: Using pictures to do computations can be dangerous as illustrated by the following example: consider the cobordism $X := W^{\mathrm{op}} \amalg W^{\mathrm{op}} \amalg W \amalg W \colon M \amalg M \rightsquigarrow M \amalg M$ (see Figure 4.5).

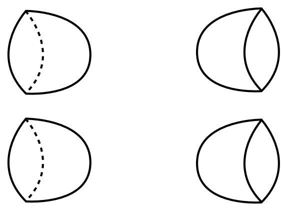


FIGURE 4.5

We then have two ways to decompose X: $(W^{\mathrm{op}} \amalg W) \amalg (W^{\mathrm{op}} \amalg W) = X = X_W \amalg W$. One might be tempted to think that then $(\mu_W, e) \sim \mathcal{S}_{X_W \amalg W} \sim \mathcal{S}_{(W^{\mathrm{op}} \amalg W) \amalg (W^{\mathrm{op}} \amalg W)} \sim (\mathrm{id}, \mathrm{id})$ implying that $\mathcal{R}^+(M)$ is contractible. However, this computation is however wrong, as one needs to consider the tangential 2-type of the outgoing boundary which is not connected in this case. $\mathcal{S}_X$ is obtained by making the structure map of X 2-connected and requiring that the structure map restricts to the one on the boundaries. Since both boundaries are the same (as θ-manifolds), we deduce that the inclusion of both the incoming and the outgoing boundary are both 2-connected. So, the obtained cobordism $\tilde{X}$ must have two components with one incoming and one outgoing boundary each. Hence, $\mathcal{S}_X \not\sim \mathcal{S}_{X_W \amalg W} \sim (\mu_W, e)$. However: If the outgoing boundary is connected, so is its tangential 2-type $\theta\colon B \to BO(d)$ and one does not have to worry about path components of the cobordism.

Corollary 4.3.3. *$\pi_0(\mathcal{R}^+(M))$ is an abelian monoid and $\pi_1(\mathcal{R}^+(M), e_W)$ is an abelian groups.*

Example 4.3.4. By going through the definition of $\mathcal{S}$ we deduce that for the case $M = S^{d-1}$ and $W = D = D^d$ we have $e_D = g_\circ^{d-1}$.

4.3.2 Dependence on W

Let us analyze how this H-space structure depends on the nullbordism $W\colon \emptyset \rightsquigarrow M$ next. From now on the symbol "=" will denote equality in the homotopy category, i.e. $f = f'$ means f and f' are homotopic.

Lemma 4.3.5. *Let $U\colon M \rightsquigarrow M$ be a θ-cobordism and let $\mathcal{S}_U\colon \mathcal{R}^+(M) \to \mathcal{R}^+(M)$ be the corresponding homotopy equivalence. Then*

$$\mu_W \circ (\mathcal{S}_U, \mathrm{id}) = \mu_W(\mathrm{id}, \mathcal{S}_U) = \mathcal{S}_U \circ \mu_W.$$

Proof. Since $W^{\mathrm{op}} \amalg W$ is bordant to $M \times I$, this lemma follows from Figure 4.6. □

Let us give an immediate application.

Corollary 4.3.6. *Let $G \subset \pi_0(\mathcal{R}^+(M))$ be the group of invertible elements with respect to μ_W. Then for any θ-cobordism $U\colon M \rightsquigarrow M$ we have $\mathcal{S}_U(G) = G$.*

Proof. Let $g, g' \in G$ such that $\mu_W(g, g') = e_W$ and let $g'' \in \pi_0(\mathcal{R}^+(M))$ such that $\mathcal{S}_U(g'') = g'$. Then by Lemma 4.3.5

$$\mu_W(\mathcal{S}_U(g), g'') = \mu_W(g, \mathcal{S}_U(g'')) = \mu_W(g, g') = e.$$

So, $\mathcal{S}_U(g)$ is a unit and $\mathcal{S}_U(G) \subset G$. The other inclusion follows analogously: $g = \mathcal{S}_U(\mathcal{S}_U^{-1}(g)) = \mathcal{S}_U(\mathcal{S}_{U^{\mathrm{op}}}(g))$ and by the above computation, $\mathcal{S}_{U^{\mathrm{op}}}(G) \subset G$. □

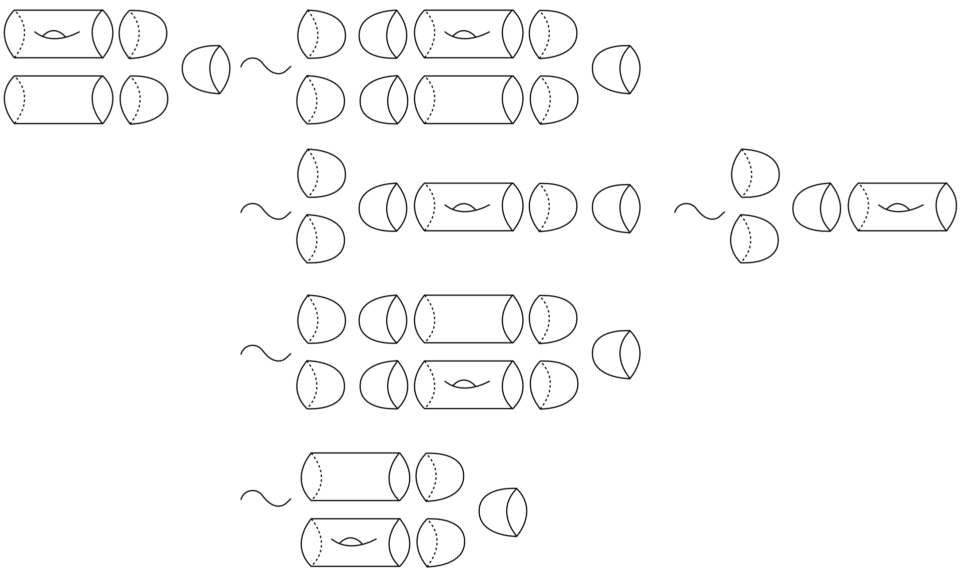


FIGURE 4.6

Now, let us analyze the dependence of this H-space structure on $W\colon \emptyset \rightsquigarrow M$. Let $V\colon \emptyset \rightsquigarrow M$ be another θ-nullbordism. We get a θ-cobordism $W^{\mathrm{op}} \amalg V\colon M \rightsquigarrow M$ and a corresponding surgery map $f := \mathcal{S}_{W^{\mathrm{op}} \amalg V}$.

Theorem 4.3.7.

1. *The map $f\colon (\mathcal{R}^+(M), \mu_W) \to (\mathcal{R}^+(M), \mu_V)$ is an equivalence of H-spaces.*
2. *We have $\mu_W = f \circ \mu_V$ and $e_V = f(e_W)$.*
3. *If $G_W, G_V \subset \pi_0(\mathcal{R}^+(M))$ denotes the respective set of units, we have $G_W = G_V$.*

Proof. 1. By Corollary 3.1.6 we have $\mathcal{S}_{V^{\mathrm{op}} \amalg W} \circ \mathcal{S}_{W^{\mathrm{op}} \amalg V} \sim \mathrm{id}$ and so f is a homotopy equivalence. For the homomorphism property we have the following computation:

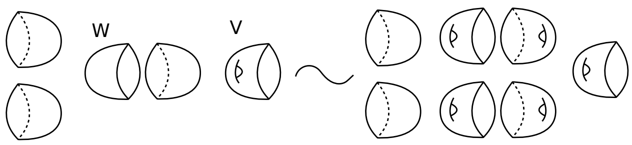


FIGURE 4.7: f is a homomorphism

2. The first part follows from Figure 4.8

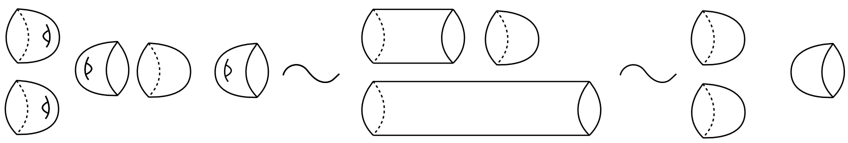


FIGURE 4.8

and the second part is handled by Figure 4.9.

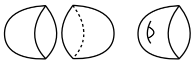


FIGURE 4.9

3. For symmetry reasons it suffices to show that every μ_W-unit is a μ_V-unit. So, let $g \in G_W$. Then $f(g) \in G_V$ because f is a homomorphism and because $f(e_W) = e_V$ and by Corollary 4.3.6 we have $f(g) \in G_V \iff g \in G_V$. □

The same proof also shows that this H-space structure is also independent of M: Let M' be another $(d-1)$-manifold with the same stabilized tangential 2-type. If $V\colon \emptyset \rightsquigarrow M'$ is a θ-nullbordism, then the map $\mathcal{S}_{W^{\mathrm{op}} \amalg V}\colon (\mathcal{R}^+(M), \mu_W) \to (\mathcal{R}^+(M'), \mu_V)$ is an equivalence of H-spaces.

4.4 Triviality and non-triviality criteria for the action map

In Subsection 4.1.2 we showed that for certain manifolds the mapping class group acts trivially on the space of psc-metrics in the homotopy category. All those manifolds had the property that *every* mapping torus was (rationally) nullbordant, no facts about the diffeomorphism itself were needed there. In this section we first give criteria for the action map to be trivial or nontrivial which have less restrictions on the manifolds they apply to but require more knowledge about the action. The first one is a criterion for the action map to be trivial which is proven using the H-space structure from the previous section. Afterwards we derive a non-triviality criterion by an argument in the style of [Car88] (cf. [Wal11, Example 1.1]). As an application we get a full characterization for the action of $\mathrm{Diff}^+(M)$ on $\mathcal{R}^+(M)$ for simply connected spin-7-manifolds.

Theorem 4.4.1. *Let $d \geq 7$ and let M^{d-1} be a simply connected* Spin*-manifold which is* Spin*-nullbordant. Let A^d be a closed* Spin*-manifold. Then $\mathcal{SE}(A) = \mathcal{S}_{M\times I \amalg A}\colon \mathcal{R}^+(M) \to$*

$\mathcal{R}^+(M)$ is homotopic to the identity if and only if $\mathcal{S}_{S^{d-1}\times I \amalg A}(g_\circ)$ and $g_\circ$ are homotopic in $\mathcal{R}^+(S^{d-1})$.

Remark 4.4.2. In particular, for $[\psi] = [f, L] \in \Gamma^{\mathrm{Spin}}(M)$ the map $f^*\colon \mathcal{R}^+(M) \to \mathcal{R}^+(M)$ is homotopic to the identity if $\mathcal{S}_{S^{d-1}\times I \amalg T_\psi}(g_\circ) \sim g_\circ$.

Proof of Theorem 4.4.1. Let $W\colon \emptyset \rightsquigarrow M$ and let $D\colon \emptyset \rightsquigarrow S^{d-1}$ denote the standard d-disc. By Theorem 4.3.7 the map $f := \mathcal{S}_{D^{\mathrm{op}} \amalg W}\colon (\mathcal{R}^+(S^{d-1}), \mu_D) \to (\mathcal{R}^+(M), \mu_W)$ is an equivalence of H-spaces. By Example 4.3.4 the neutral element e_D is given by the round metric $g_\circ$ and $f(g_\circ) \sim f(e_D) \sim e_W$ by Theorem 4.3.7. We compute

$$\begin{aligned}\mathcal{SE}_A(\_) = \mathcal{S}_{M\times I \amalg A} &= \mu_W(\mathcal{S}_{M\times I \amalg A}(\_), e_W) = \mu_W(\_, \mathcal{S}_{M\times I \amalg A}(e_W)) \\ &= \mu_W(\_, \mathcal{S}_{M\times I \amalg A} \circ f(g_\circ)) = \mu_W(\_, f \circ \mathcal{S}_{S^{d-1}\times I \amalg A}(g_\circ)),\end{aligned}$$

where the last equality follows from the Figure 4.10.

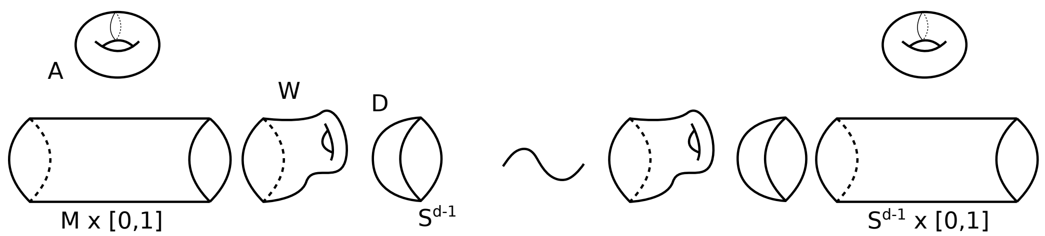


FIGURE 4.10

Now, $\mu_W(\_, f \circ \mathcal{S}_{S^{d-1}\times I \amalg A}(g_\circ)) = \mathrm{id}$ if and only if $f \circ \mathcal{S}_{S^{d-1}\times I \amalg A}(g_\circ) \sim e_W$ which happens if and only if $\mathcal{S}_{S^{d-1}\times I \amalg A}(g_\circ) \sim g_\circ$. □

Next we give the non-triviality criterion.

Proposition 4.4.3. *Let M be a $(d-1)$-dimensional, simply connected* Spin*-manifold and let W^d be a closed* Spin*-manifold with $\hat{A}(W) \neq 0$. Then $\mathcal{SE}_W(g) \not\sim g$ for every psc-metric g on M. In particular, $\mathcal{SE}_W$ is not homotopic to the identity.*

Proof. By Lemma B.4 we can perform (Spin-)surgery on $M \times [0,1] \amalg W$ to get an admissible cobordism $V\colon M \rightsquigarrow M$. If $\mathcal{SE}_W$ is homotopic to the identity there exists a psc-metric G on V that restricts to the same metric g_0 on both boundaries by Lemma 1.7.12 (see Remark 3.1.7 as well). We obtain a psc-metric on the manifold V' obtained by gluing the boundaries of V together along the identity. So, $\hat{A}(V') = 0$ by the Lichnerowicz-formula and since $\hat{A}$ is Spin-cobordism invariant we get

$$0 = \hat{A}(V') = \hat{A}(M \times S^1 \amalg W) = \hat{A}(W). \qquad \square$$

We can now derive the result for 7-manifolds.

Corollary 4.4.4. *Let M^7 be a simply connected* Spin*-manifold and let $f\colon M \xrightarrow{\cong} M$ be a* Spin*-diffeomorphism. Then the following are equivalent:*

1. *$\hat{A}(T_f) = 0$.*
2. *T_f is* Spin *nullbordant.*
3. *f^* is homotopic to the identity.*
4. *$f^*g \sim g$ for every $g \in \mathcal{R}^+(M)$.*
5. *There exists a metric $g \in \mathcal{R}^+(M)$ such that $f^*g \sim g$.*

Proof. The implications $3. \Rightarrow 4.$ and $4. \Rightarrow 5.$ are obvious and the implication $2. \Rightarrow 3$ follows from Corollary 4.1.1. For $1. \Rightarrow 2.$ we note that

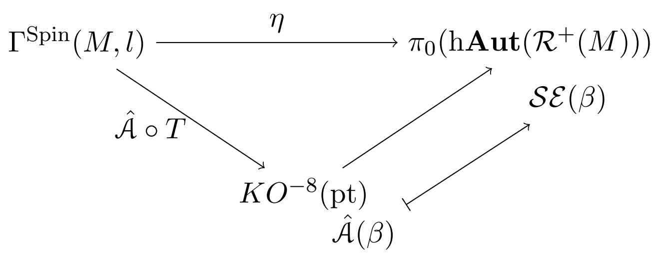
$$\Omega_8^{\mathrm{Spin}} \cong \mathbb{Z} \oplus \mathbb{Z} \cong \langle [\mathbb{HP}^2], [\beta] \rangle,$$

where β denotes the Bott manifold with $\hat{A}(\beta) = 1$ and $\mathrm{sign}(\beta) = 0$. Furthermore, $\mathrm{sign}(\mathbb{HP}^2) \neq 0$ and $\hat{A}(\mathbb{HP}^2) = 0$. Since for T_f both these invariants vanish, it has to be Spin-nullbordant. Finally $5. \Rightarrow 1.$ is proven as follows: Let g_t be an isotopy between f^*g and g. Since isotopy of psc metrics implies concordance of psc metrics, there exists a psc metric G on $M \times [0,1]$ restricting to f^*g and g. Then G induces a psc metric on T_f as one can identify the metrics on the boundary along f^* and hence $\hat{A}(T_f) = 0$. $\square$

Remark 4.4.5.

1. Since M is simply connected we have $\mathrm{Diff}^{\mathrm{Spin}}(M) \twoheadrightarrow \mathrm{Diff}^+(M)$. Hence the above Corollary classifies the action of $\Gamma^+(M)$ on $\mathcal{R}^+(M)$ for every simply connected 7-dimensional Spin-manifold.
2. Note that the implication $5. \Rightarrow 1.$ does not require the restriction to dimension 7.

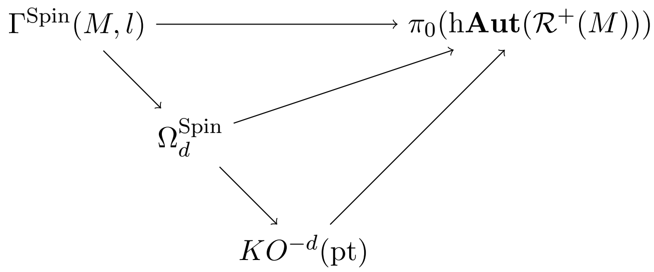
In the 7-dimensional case we get a further factorization of the action map:



This factorization is sharp in the sense that $\ker \eta = \ker \hat{\mathcal{A}} \circ T$. We close this chapter with 2 questions:

Question 4.4.6. Let M be simply connected and spin. Is vanishing of the $\hat{\mathcal{A}}$-genus of W a sufficient condition for $\mathcal{SE}_W$ to be homotopic to the identity on $\mathcal{R}^+(M)$?

If the answer to Question 4.4.6 were yes, we get the following commutative extension of the diagram above.



In the 7-dimensional case, this would be implied by the following conjecture.

Conjecture 4.4.7. $\mathcal{SE}(\mathbb{HP}^2) \sim \mathrm{id}$.

The motivation for the second question is the fact that the kernel of the signature homomorphism $\mathrm{sign}\colon \Omega_*^{\mathrm{SO}} \to \mathbb{Z}$ is generated by mapping tori, i.e. for every $W \in \ker \mathrm{sign}$ there exists an oriented manifold M^{d-1} and an orientation preserving diffeomorphism $f \in \mathrm{Diff}^+(M)$ such that $[T_f] = [W] \in \Omega_d^{\mathrm{SO}}$ (cf. [Win71]). We consider the group homomorphisms

$$\Omega_d^{\mathrm{Spin}} \longrightarrow \Omega_d^{\mathrm{SO}} \overset{\mathrm{sign}}{\longrightarrow} \mathbb{Z}$$

and we ask the following question.

Question 4.4.8. Which elements $W \in \ker \mathrm{sign}$ with nontrivial $\hat{A}$-genus are in the image of the homomorphism $\Omega_d^{\mathrm{Spin}} \longrightarrow \Omega_d^{\mathrm{SO}}$, i. e. can be represented by the mapping torus of a Spin-diffeomorphism on a Spin-manifold?

If M were a simply connected Spin-manifold of positive scalar curvature with a Spin-diffeomorphism (f, L) whose mapping torus has non-vanishing $\hat{A}$-genus, we would get a detection result for every curvature condition that implies positive scalar curvature by Proposition 4.4.3.

5

Surgery stable curvature conditions

In this chapter we generalize our main result to other curvature conditions. The key observation is that the two main (differential) geometric ingredients that go into the proof are: The surgery theorem (1.7.8) by Chernysh [Che04b, Theorem 1] and [Wal11, Lemma 1.9] which says that double torpedo metrics and mixed torpedo metrics on the sphere lie in the component of the round metric. We call this second property the *mixed torpedo condition*. Recent work of Kordass [Kor18] improves Chernysh's result from positive scalar curvature to other curvature conditions[1]. Afterwards we use this improved surgery theorem to indicate how one can upgrade a detection result of Botvinnik–Ebert–Randal-Williams [BERW17].

5.1 The improved surgery theorem

Let us fix $c \geq 3$ and let C be a *deformable, codimension c surgery stable curvature condition of dimension* $(d-1)$.[2] We denote by $\mathcal{R}_C(M)$ the *space of Riemannian metrics satisfying* C. Note that saying a metric satisfies C only makes sense for metrics on manifolds

[1] This is built on work of Hoelzel [Hoe16] in the same way Chernysh's result [Che04b] is built on Gromov-Lawson's surgery theorem [GL80].

[2] For a definition and examples see [Kor18, Section 2.1] and [Hoe16, Introduction].

of dimension $(d-1)$. The round metric $g_\circ^{k-1}$ of radius 1 satisfies $g_\circ^{k-1} + g_{eucl} \in \mathcal{R}_C(S^{k-1} \times \mathbb{R}^{d-k})$ for ε small enough and $k \geq c$ (cf. [Kor18, Remark 2.8]) and for every metric g_N on a manifold N^{k-1} there exists a torpedo metric $g_{tor}^{d-k} \in \mathcal{R}(D^{d-k})_{g_\circ}$ such that $g_N + g_{tor} \in \mathcal{R}_C(N \times D^k)_{g_N + g_\circ}$ provided that $d - k \geq 3$ (cf. [Kor18, Corollary 2.27]). We define

$$\mathcal{R}_C(M, \varphi) := \{g \in \mathcal{R}_C(M) \colon \varphi^* g = g_\circ + g_{tor}\}$$

The following is the improved parametrized surgery theorem.

Theorem 5.1.1 ([Kor18, Theorem 3.5]). *Let C be a deformable, codimension c surgery stable curvature condition, M be a $(d-1)$-manifold and let $\varphi \colon S^{k-1} \times D^{d-k} \hookrightarrow M$ be a surgery datum with $k \leq d - c$. Then the inclusion*

$$\mathcal{R}_C(M, \varphi) \hookrightarrow \mathcal{R}_C(M)$$

is a (weak) homotopy equivalence.

Example 5.1.2. Let us list some curvature conditions to which Theorem 5.1.1 applies.

1. Positive scalar curvature is a deformable, codimension 3 surgery stable curvature condition, so Kordass' result really is a generalization of Chernysh's theorem.
2. Let (M, g) be a Riemannian manifold and let $P \subset T_x M$ be a p-dimensional subspace with (E_i) an orthonormal basis of $P^\perp$. We define the *p-curvature of g* by $s_p(g)(P) := \sum_{i,j=1}^{n-p} \mathbf{sec}_g(\langle E_i, E_j \rangle)$.[3] Then *positive p-curvature* is a deformable, codimension $p + 3$-surgery stable curvature condition (see [Lab97] and [Kor18, Example 2.20]).
3. In a similar fashion, one can define q-Ricci curvature. Let $Q \subset T_x M$ be a q-dimensional subspace with (E_i) as an orthonormal basis. We define the *q-Ricci-curvature of g* by $\mathbf{Ric}_q(g)(Q) = \sum_{i=1}^{q} \mathbf{Ric}(g)(E_i)$.[4] Then, for $2 \leq q \leq d-2$, *positive q-Ricci curvature* is a deformable, codimension $(d - q + 1)$-surgery stable curvature condition (see [Wol12] and [Kor18, Example 2.20]).

All these examples are contained in positive scalar curvature, so for all these conditions one has $\mathcal{R}_C(M) \subset \mathcal{R}_{psc}(M) =: \mathcal{R}^+(M)$.

It is not clear if deformable, codimension c surgery stable curvature conditions also *encode the mixed torpedo condition*, i. e. that every metric satisfying C also satisfies the

[3] Clearly, 0-curvature is scalar curvature and $(d-2)$-curvature is the sectional curvature. We also see that $s_1(g)(P) = \mathrm{scal} g - 2\mathrm{Ric}(P)$.

[4] $(d-1)$-Ricci-curvature is scalar curvature and 1-Ricci curvature is ordinary Ricci curvature.

mixed torpedo condition. For the above examples it should be true and we will assume it for the succeeding section.

5.2 Generalization of Theorem 3.3.1

In this section we generalize our main Theorem 3.3.1. All our results carry over without any change in the proofs. However, the dimension restriction changes. Let $m := \max\{c+4, 2c\}$.

Remark 5.2.1. With the same proof as in Theorem 1.6.2 one can show the following: Let $a, b \in \{-1, 0, \dots, d\}$, W be of dimension at least $d \geq \max\{a+5, b+5, a+b+2\}$ and let the inclusions (W, M_0) and (W, M_1) be a- and b-connected, respectively. Then $\mathcal{H}_{a+1,d-b-1}(W)$ is path-connected.

Theorem 5.2.2 (cf. Theorem 2.3.3). *For $d \geq m(c)$, the functor $\mathcal{P}^{-1,c-1}$ is an equivalence of categories.*

From now on let $c \geq 3$ and let us fix be a deformable codimension c surgery stable curvature condition C that encodes the mixed torpedo condition.

Lemma 5.2.3 (cf. Definition 3.1.1, Lemma 3.1.3 and Corollary 3.1.5). *Let $d \geq m(c)$. Then there is a functor*

$$\overline{\mathcal{S}}^C : \mathcal{B}ord_d^{-1,c-1} \longrightarrow \mathrm{hTop}$$

which satisfies:

1. $\overline{\mathcal{S}}^C(M) = \mathcal{R}^+(M)$
2. $\overline{\mathcal{S}}^C_{(M\times I, \mathrm{id}, f)} = f_*$
3. $\overline{\mathcal{S}}^C_{\boldsymbol{tr}\,\varphi, \mathrm{id}, \mathrm{id}}(g) = \overline{\mathcal{S}}^C_{\varphi}$.

Lemma 5.2.4 (cf. Lemma 3.2.1). *Let $d \geq 2c+1$ and let M_0, M_1 be two $(d-1)$-manifolds, let $W = [W, \mathrm{id}, \mathrm{id}] \in \boldsymbol{mor}_{\mathcal{B}\mathrm{ord}_d^{-1,c-1}}(M_0, M_1)$ and let $\Phi \colon S^{k-1} \times D^{d-k+1} \hookrightarrow \mathrm{Int}\, W$ be an embedding with $c \leq k \leq d-c$. Then $\overline{\mathcal{S}}^C_W \sim \overline{\mathcal{S}}^C_{W_\Phi}$.*

Now, let $\theta \colon B \to BO(d)$ be a fibration which is once-stable. Let $\hat{\Omega}^\theta_{d,c-1}$ denote the category which has $(d-1)$-dimensional θ-manifolds $(M^{d-1}, \hat{l})$ as objects and let the set of morphisms from M_0 to M_1 be given by $\Omega^\theta_d(M_0, M_1)$ if the underlying structure map $l \colon M_1 \to B$ is $(c-1)$-connected and be empty otherwise.

Theorem 5.2.5 (cf. Theorem 3.3.1)**.** *Let $d \geq 2c+1$. Then there is a unique functor*

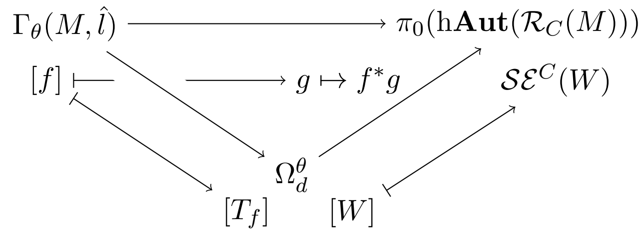
$$\mathcal{S}^C : \hat{\Omega}^\theta_{d,c-1} \longrightarrow \mathrm{hTop}$$

such that

1. $\mathcal{S}^C(M) = \mathcal{R}_C(M)$,
2. $\mathcal{S}^C(M \times I, \mathrm{id}, f^{-1}) = [g \mapsto f^* g]$,
3. $\mathcal{S}^C(\boldsymbol{tr}\,\varphi, \mathrm{id}, \mathrm{id}) = \overline{\mathcal{S}}^C_\varphi$.

Remark 5.2.6. One can possibly improve the dimension restriction from $d \geq 2c+1$ to $d \geq m(c)$. The 2-index theorem only requires $d \geq m(c)$ but for the use of Lemma A.1 in the proof of Lemma 5.2.4 (cf. Lemma 3.2.1) one needs $d \geq 2c+1$. It is possible that Lemma A.1 also holds for one dimension smaller (cf. Remark A.2.4). This would mean that the restriction becomes $d \geq 7$ for $c = 3$ and $d \geq 2c$ for $c \geq 4$.

Corollary 5.2.7 (cf. Corollary 4.1.1)**.** *Let $d \geq 2c+1$ and let $\theta \colon B \to B\mathrm{O}(d)$ be the stabilized tangential $(c-1)$-type of M^{d-1}. Then there is a group homomorphism $\mathcal{SE}^C : \Omega^\theta_d \to \pi_0\big(\mathrm{h}\mathbf{Aut}(\mathcal{R}_C(M))\big)$ such that the following diagram commutes:*



Some of the examples computed in Section 4.1 have analogues for other deformable codimension c surgery stable curvature conditions. Let $\Omega^{\langle k \rangle}_d$ denote the cobordism group of d-dimensional $B\mathrm{O}\langle k \rangle$-manifolds. The key observation is the following:

Proposition 5.2.8. *The forgetful map $\Omega^{\langle k \rangle}_d \otimes \mathbb{Q} \longrightarrow \Omega^{\langle 1 \rangle}_d \otimes \mathbb{Q} = \Omega^{\mathrm{SO}}_d \otimes \mathbb{Q}$ is injective for every $k \geq 2$.*

Proof. In this proof $H_n(X)$ denotes the *rational* homology of X. By [KL05, Theorem 2.1] there is an isomorphism $\Omega^{\langle k \rangle}_* \otimes \mathbb{Q} \xrightarrow{\cong} H_*(B\mathrm{O}\langle k \rangle)$. We show that the map $B\mathrm{O}\langle l \rangle \longrightarrow B\mathrm{O}\langle l-1 \rangle$ induces a monomorphism in rational homology for $2 \leq l \leq k$. The homotopy groups of $B\mathrm{O}$ are given by

$$\pi_i(B\mathrm{O}) = \begin{cases} \mathbb{Z} & \text{if } i \equiv 0(4) \\ \mathbb{Z}/2\mathbb{Z} & \text{if } i \equiv 1,2(8) \\ 0 & \text{otherwise.} \end{cases}$$

So, $B\mathrm{O}\langle l\rangle = B\mathrm{O}\langle l-1\rangle$ for $l \equiv 3,5,6,7(8)$ and we get fibrations

$$B\mathrm{O}\langle 4m\rangle \longrightarrow B\mathrm{O}\langle 4m-1\rangle \longrightarrow K(\mathbb{Z}, 4m)$$
$$B\mathrm{O}\langle 8m+e\rangle \longrightarrow B\mathrm{O}\langle 8m+e-1\rangle \longrightarrow K(\mathbb{Z}/2\mathbb{Z}, 8m+e)$$

for $e = 1,2$. The rational cohomology of $K(\mathbb{Z}/2\mathbb{Z}, 8m+e)$ vanishes. The base space of these fibrations is simply connected by our assumption on l and so by the Leray–Serre spectral sequence the map $B\mathrm{O}\langle l\rangle \longrightarrow B\mathrm{O}\langle l-1\rangle$ induces an isomorphism in rational homology unless $l = 4m$. If $l = 4m$ this spectral sequence gives $E^2_{p,q} = H_p(K(\mathbb{Z}, 4m)) \otimes H_q(B\mathrm{O}\langle 4m\rangle) \Rightarrow H_{p+q}(B\mathrm{O}\langle 4m-1\rangle)$. The entries on the E^2-page are zero if either p or q is not divisible by 4. Therefore every differential of the spectral sequence is trivial and the spectral sequence collapses at the E^2-page. We therefore have

$$H_n(B\mathrm{O}\langle 4m\rangle) \hookrightarrow \bigoplus_{p+q=n} H_p(K(\mathbb{Z}, 4m)) \otimes H_q(B\mathrm{O}\langle 4m\rangle) \xrightarrow{\cong} H_n(B\mathrm{O}\langle 4m-1\rangle) \qquad \square$$

Example 5.2.9. For $c = 3,4$, we have $B\mathrm{O}(d)\langle c-1\rangle = B\mathrm{Spin}(d)$ and for $5 \leq c \leq 8$, $B\mathrm{O}(d)\langle c-1\rangle = B\mathrm{String}(d)$.

We can now state analogues for two examples from Section 4.1.

Corollary 5.2.10. *Let $d \geq 2c+1$ and let M^{d-1} be a $c-2$-connected $B\mathrm{O}(d)\langle c-1\rangle$ manifold. If $d \equiv 0(4)$ let all Pontryagin classes of M vanish. Let $f\colon M \xrightarrow{\cong} M$ be an orientation preserving diffeomorphism. Then $(f^*)^n\colon \mathcal{R}_C(M) \to \mathcal{R}_C(M)$ is homotopic to the identity for some $n \in \mathbb{N}$.*

Corollary 5.2.11. *Let $d \geq 2c+1$ and let M^{d-1} be a $B\mathrm{O}(d)\langle c-1\rangle$ manifold with finite fundamental group and $c-2$-connected universal cover. If $d \equiv 0(4)$ let all Pontryagin classes of M vanish. Let $f\colon M \xrightarrow{\cong} M$ be a $B\mathrm{O}(d)\langle c-1\rangle$-diffeomorphism acting on the fundamental group by an inner automorphism. Then $(f^*)^n$ is homotopic to the identity for some $n \in \mathbb{N}$.*

For $c = 3, 4$ we have $BO(d)\langle c-1\rangle = B\mathrm{Spin}(d)$ and even more results from Chapter 4 carry over. If $c = 3$ this works for all results and if $c = 4$ one has to consider 2-connected manifolds instead of simply connected manifolds.

5.3 A detection result for $\mathcal{R}_C(M)$

In this section we use Kordass' Theorem 5.1.1 to indicate how one can generalize the work of Botvinnik, Ebert and Randal-Williams [BERW17]. Let $c \in \{3, \ldots, d-3\}$ and let C be a deformable, codimension c surgery stable curvature condition that implies positive scalar curvature. Also, let $\iota\colon \mathcal{R}_C(M) \hookrightarrow \mathcal{R}^+(M)$ denote the inclusion. The following Lemma states the existence of stable metrics in a special case.

Lemma 5.3.1 ([BERW17, Theorem 2.6]). *Let $d \geq 2c$ and let $V^{d-1}\colon S^{d-2} \rightsquigarrow S^{d-2}$ be a $(c-2)$-connected, $BO(d)\langle c-1\rangle$-cobordism. Also, assume that V is $BO(d)\langle c-1\rangle$-cobordant to $S^{d-2} \times [0,1]$ relative to the boundary. Then there exists a metric $g \in \mathcal{R}_C(V)_{g_\circ, g_\circ}$ with the following property: If $W : S^{d-2} \rightsquigarrow S^{d-2}$ is cobordism and $h \in \mathcal{R}(S^{d-2})$ is a boundary condition such that $h + dt^2 \in \mathcal{R}_C(S^{d-2} \times [0,1])$ then the two gluing maps*

$$\mu(\_, g)\colon \mathcal{R}_C(W)_{h,g_\circ} \longrightarrow \mathcal{R}_C(W \cup V)_{h,g_\circ}$$
$$\mu(g, \_)\colon \mathcal{R}_C(W)_{g_\circ,h} \longrightarrow \mathcal{R}_C(V \cup W)_{g_\circ,h}$$

are homotopy equivalences.

Proof. By assumption there exists a relative $BO(d)\langle c-1\rangle$-cobordism $X\colon V \rightsquigarrow S^{d-2} \times [0,1]$ and by performing surgery on the interior of X we may assume X has no handles of indices $0, \ldots, c-1, d-c+1, \ldots, d$ (cf. Lemma B.4 and Lemma 1.6.5). So $S^{d-2} \times [0,1]$ is obtained from V by a sequences of surgeries with these indices. Let $\varphi\colon S^{k-1} \times D^{d-k} \hookrightarrow V$ be such a surgery embedding with $k \in \{c, \ldots, d-c\}$ and let $g \in \mathcal{R}_C(V)_{g_\circ,g_\circ}$ and let $g' \in [\overline{\mathcal{S}}^C_\varphi(g)] \in \pi_0(\mathcal{R}_C(V_\varphi))$. The map $\overline{\mathcal{S}}^C_\varphi$ is a homotopy equivalence and so gluing on the metric g is a homotopy equivalence if and only if gluing on g' is a homotopy equivalence.

Now gluing in $(S^{d-2} \times [0,1]), g_\circ + dt^2)$ is a homotopy equivalence and so by the above argument there exists a metric $g \in \mathcal{R}_C(V)_{g_\circ,g_\circ}$ as required. □

Let M^{d-1} be a manifold with boundary ∂M and let $h \in \mathcal{R}(\partial M)$ such that $h + dt^2 \in \mathcal{R}_C(\partial M \times [0,1])$. The space $\mathcal{R}_C(M)_h$ has an action of $\mathrm{Diff}_\partial(M)$, the group of diffeomorphisms which are the identity on a neighbourhood of ∂M. We get an action map $\eta^C \colon \mathrm{Diff}_\partial(M) \to \mathrm{h}\mathbf{Aut}(\mathcal{R}_C(M)_h)$ which induces

$$\Gamma_\partial(M) := \pi_0(\mathrm{Diff}_\partial(M)) \longrightarrow \pi_0(\mathrm{h}\mathbf{Aut}(\mathcal{R}_C(M)_h)) \tag{5.1}$$

The psc-analog of the following is one of the main ingredients in the proof of [BERW17, Theorem B].

Theorem 5.3.2 ([BERW17, Theorem 4.1])**.** *Let $d \geq 2c$ and let M^{d-1} be a $(c-2)$-connected, $B\mathrm{O}(d)\langle c-1\rangle$-manifold with boundary $\partial M = S^{d-2}$. Also, assume that M is $B\mathrm{O}(d)\langle c-1\rangle$-cobordant to D^{d-1} relative to the boundary. Then the image of the map (5.1) for $h = g_\circ^{d-2}$ is an abelian group.*

For the proof we will use the following Lemma of Eckmann-Hilton style.

Lemma 5.3.3 ([BERW17, Lemma 4.2])**.** *Let $\mathcal{C}$ be a nonunital topological category with objects the integers and let G be a topological group which acts on $\mathcal{C}$, i.e. G acts on all morphism spaces and the composition in $\mathcal{C}$ is G-equivariant. We will denote the composition of x and y by $x \cdot y$. Suppose that*

1. *$\mathcal{C}(m,n) = \emptyset$ for $n \leq m$.*
2. *For each $m \neq 0$ there exists a $u_m \in \mathcal{C}(m, m+1)$ such that the composition maps*

$$u_m \cdot \_ \colon \mathcal{C}(m+1,n) \to \mathcal{C}(m,n) \qquad \text{for } n > m+1$$
$$\_ \cdot u_m \colon \mathcal{C}(n,m) \to \mathcal{C}(n,m+1) \qquad \text{for } n < m$$

   are homotopy equivalences.
3. *There exists an $x_0 \in \mathcal{C}(0,1)$ such that the composition maps*

$$x_0 \cdot \_ \colon \mathcal{C}(1,n) \to \mathcal{C}(0,n) \qquad \text{for } n > 1$$
$$\_ \cdot x_0 \colon \mathcal{C}(n,0) \to \mathcal{C}(n,1) \qquad \text{for } n < 0$$

   are homotopy equivalences.
4. *The G-action is trivial unless $m \leq 0$ and $1 \leq n$.*

Then for $f, g \in G$ the maps $f, g \colon \mathcal{C}(0,1) \to \mathcal{C}(0,1)$ commute up to homotopy.

Proof of Theorem 5.3.2. This is completely analogous to [BERW17, Proof of Theorem 4.1]. Consider a closed disk $D \subset S^{d-2} \times (0,1)$. By Theorem 5.1.1 there exists a metric $h \in \mathcal{R}_C(S^{d-2} \times [0,1], D; g_{tor}^{d-1})_{g_\circ^{d-2}, g_\circ^{d-2}}$ which is isotopic to the product metric $g_\circ^{d-2} + dt^2$ relative to the boundary. By cutting out this disk we obtain a metric $\overline{h}$ on $T := (S^{d-2} \times [0,1]) \setminus \operatorname{int}(D)$ that satisfies C. We denote by $P = S^{d-2}$ the boundary component created by cutting out this disk. We get the composition

$$\mathcal{R}_C(M)_{g_\circ} \xrightarrow{\mu_{\overline{h}}} \mathcal{R}_C(M \cup_{S^{d-2}\times\{0\}} T)_{g_\circ, g_\circ} \xrightarrow{\mu_{g_{tor}}} \mathcal{R}_C(M \cup_{S^{d-2}\times\{0\}} T \cup_P D)_{g_\circ}$$

given by gluing in $(T, \overline{h})$ and (D, g_{tor}). The composition is given by gluing in h which is homotopic to gluing in $g_\circ + dt^2$ and so it is a homotopy equivalence. The right-most map is a homotopy equivalence by Theorem 5.1.1 and so $\mu_{\overline{h}}$ also is a homotopy equivalence. Let $V := M \cup_{S^{d-2}\times\{0\}} T$ and let us consider this as a cobordism $S^{d-2} = P \rightsquigarrow S^{d-2} \times \{1\} = S^{d-2}$.

We now apply Lemma 5.3.3 to the following scenario: Let $G := \operatorname{Diff}_\partial(M)$ and let $\mathcal{C}(0,1) = \mathcal{R}_C(V)_{g_\circ, g_\circ}$. Furthermore, let

$$\mathcal{C}(m,n) = \begin{cases} \mathcal{R}_C(S^{d-2} \times [m,0] \cup V \cup S^{d-2} \times [0,n])_{g_\circ, g_\circ} & \text{for } m \le 0, n \ge 1 \\ \mathcal{R}_C(S^{d-2} \times [m,n])_{g_\circ, g_\circ} & \text{for } m < n \le 0 \text{ or } n > m \ge 1 \\ \emptyset & \end{cases}$$

Let G act on $\mathcal{C}(m,n)$ by extending a diffeomorphism $f \in \operatorname{Diff}_\partial(M)$ by the identity and then acting via pullback, i. e. G acts on M via pullback and trivially everywhere else. With this action the composition given by gluing metrics is obviously G-equivariant. For $m \neq 0$ let $u_m := g_\circ^{d-2} + dt^2 \in \mathcal{C}(m, m+1)$ and by Lemma 5.3.1 there exists an $x_0 \in \mathcal{C}(0,1)$ such that the hypothesis of Lemma 5.3.3 is satisfied and so the action of $\operatorname{Diff}_\partial(M)$ on $\mathcal{R}_C(V)_{g_\circ, g_\circ}$ factors through an abelian group. The Theorem follows because the gluing map $\mu_{\overline{h}}\colon \mathcal{R}_C(M)_{g_\circ} \to \mathcal{R}_C(V)_{g_\circ, g_\circ}$ is a $\operatorname{Diff}_\partial(M)$-equivariant homotopy equivalence. □

Remark 5.3.4. This can also be proven by using a version of Corollary 4.1.1 for manifolds with boundary and using Lemma 4.2.4 as in Section 4.2. This comes at the price of requiring that M is $(c-1)$-connected and that C encodes the mixed torpedo condition.

We will now give an outline how one can possibly generalize the detection result from [BERW17]. Let $\theta\colon BO(d-1)\langle c-1\rangle \to BO(d-1)$ be the $(c-1)$-connected cover. Let W_0 be a manifold of dimension $d-1 = 2n \ge 2c$ which is $BO\langle c-1\rangle$-cobordant to D^{2n}

and that satisfies that the structure map $l\colon W_0 \to BO(d-1)\langle c-1\rangle$ is $c-1$-connected. Using Theorem 5.3.2 one can construct a map $\rho\colon \Omega^{\infty+1}MT\theta \longrightarrow \mathcal{R}_C(W_0)$ in the same way as in [BERW17, Chapter 4]. For $c = 3, 4$ we have $BO(d)\langle c-1\rangle = B\mathrm{Spin}(d)$ and one should be able to adjust the index theoretic arguments from [BERW17, Chapter 3] to show that the composition

$$\Omega^{\infty+1}MT\mathrm{Spin}(d-1) \xrightarrow{\ \rho\ } \mathcal{R}_C(M)_h \hookrightarrow \mathcal{R}^+(M)_h \xrightarrow{\ \mathrm{inddiff}_{g_0}\ } \Omega^{\infty+d}KO$$

where is weakly homotopic to the loop map of $\hat{\mathcal{A}}_{d-1}$. Employing a propagation theorem in the style of [BERW17, Proposition 3.18] this can then be upgraded to hold for all Spin-manifolds of dimension $2n$.

Furthermore, if C is *stable*, it is possible to define a restriction map $\mathrm{res}\colon \mathcal{R}_C(W) \to \mathcal{R}_C(M)$ for a manifold W with boundary M. It has been told to us by J.B. Kordaß that this is a Serre-fibration as well and so the above should imply a corresponding result for odd-dimensional manifolds of dimension at least $2c+1$ (cf. [BERW17, Section 3.6]). Therefore one should get the same detection results as in [BERW17] for positive $(d-2)$-Ricci curvature on manifolds of dimension at least 6. If one considers manifolds of dimension at least 8, one should obtain these results for positive 1-curvature and positive $(d-3)$-Ricci curvature, too.

It is however unclear if the map $\mathrm{inddiff}_{g_0} \circ \rho$ can also be used to detect families of metrics satisfying a deformable codimension c surgery stable curvature condition if $c \geq 5$.

A

Multijet-transversality

In this chapter we will give a proof of the following two lemmas using Multijet-transversality.

Lemma A.1. *Let W^d, $d \geq 2k+1$ be a manifold and let $V \subset W$ be a codimension 0 submanifold such that (W, V) is $(k-1)$-connected. Then $\mathrm{Emb}(S^{k-1} \times D^{d-k+1}, V) \longrightarrow \mathrm{Imm}(S^{k-1} \times D^{d-k+1}, W)$ is a π_0-bijection.*

Lemma A.2. *Let $h_0, h_1 \colon W^d \to [0,1]$ be Morse functions. Then there exists a generic path of generalized Morse functions connecting them.*

A.1 Jet bundles

In this section we recall the notion of *jet bundles* and *jet transversality*. This is a recollection from [GG73].

Definition A.1.1 ([GG73, pp. 37]). Let M, N be smooth manifolds, $p \in M$ and let $f, g \colon M \to N$ be smooth maps with $f(p) = g(p)$. We say

1. f has *first order contact* with g at p if $\mathrm{d}f_p = \mathrm{d}g_p$ as mappings $T_pM \to T_{f(p)}N$.

2. f has *k-th order contact* with g at p if $\mathrm{d}f$ has $(k-1)$st order contact with $\mathrm{d}g$ at every point in T_pM. This shall be written as $f \sim_k g$ at p.
3. We denote by $J^k(M,N)_{p,q}$ the equivalence classes of maps $f\colon M \to N$ satisfying $f(p) = q$ under the relation $f \sim_k g$ at p.
4. We define
$$J^k(M,N) := \bigcup_{(p,q)\in M\times N} J^k(M,N)_{p,q}$$
An element of $J^k(M,N)$ is called a *k-jet* (of mappings) from M to N.
5. Let $\sigma \in J^k(M,N)$. Then there exist $(p,q) \in M \times N$ such that $\sigma \in J^k(M,N)_{p,q}$. We call p the *source* and q the *target* of σ. We define the *source map* $\alpha\colon J^k(M,N) \to M$ and *target map* $\beta\colon J^k(M,N) \to N$.
6. For $f\colon M \to N$ we denote by $j^kf\colon M \to J^k(M,N)$ the map that sends p to the equivalence class of f in $J^k(M,N)_{p,f(p)}$.

Definition A.1.2 ([GG73, Definition 4.1]). Let M, N be smooth manifolds and $f\colon M \to N$ smooth. Let X be a submanifold of N and $p \in M$. We say that f *intersects* X *transversely at* p (denoted by $f \pitchfork X$ at p) if either $f(p) \notin X$ or $f(p) \in X$ and $T_{f(p)}N = T_{f(p)}X + df_p(T_pM)$. If $A \subset M$, f is defined to intersect X *transversely on* A (denoted by $f \pitchfork X$ on A) if $f \pitchfork X$ at all $p \in A$. If $A = M$ we simply write $f \pitchfork X$.

Theorem A.1.3 ([GG73, Theorem 4.4]). *Let M, N, X, f as above and assume that $f \pitchfork X$. Then $f^{-1}(X)$ is a submanifold of M and $\operatorname{codim} f^{-1}(X) = \operatorname{codim}(X)$. In particular, if $\operatorname{codim} X = \dim M$ and M is compact, $f^{-1}(X)$ is a finite collection of points.*

Definition A.1.4 ([GG73, Definition 3.2]). Let F be a topological space. A subset G of F is *residual* if it is the countable intersection of open dense subsets. F is called a *Baire space* if every nonempty residual set is dense.

Proposition A.1.5 ([GG73, Proposition 3.3]). *$C^\infty(M,N)$ with the C^∞-topology is a Baire space.*

Definition A.1.6 ([GG73, p. 57]). Let $f\colon M \to N$, $s \in \mathbb{N}$ and let $M^{(s)} := \{(x_1,\ldots,x_s) \in M^s \mid x_i \neq x_j \text{ for } i \neq j\}$. Furthermore, let $J^k_s(M,N) := (\alpha^s)^{-1}(M^{(s)})$, the *$s$-fold k-jet bundle*, where α^s denotes the s-fold product of the map α. We define the map $j^k_sf\colon M^{(s)} \to J^k_s(M,N)$ by $j^k_sf(x_1,\ldots,x_s) := (j^kf(x_1),\ldots,j^kf(x_s))$.

Corollary A.1.7. *Let $f \in C^\infty(M,N)$ and let $X \subset J^k_s(M,N)$ be a submanifold such that $j^k_sf \pitchfork X$. Then $(j^k_sf)^{-1}(X)$ is a submanifold of $M^{(s)}$ of dimension $s \cdot \dim M - \operatorname{codim} X$.*

Theorem A.1.8 (Multijet Transversality Theorem, [GG73, Theorem 4.13])**.** *Let M, N be smooth manifolds, $X \subset J^k_s(M,N)$ be a submanifold and let*

$$T_X := \{f \in C^\infty(M,N) \mid j^k_s f \pitchfork X\}.$$

Then T_X is a residual subset of $C^\infty(M,N)$ and hence nonempty.

There also is a relative version of the Multijet Transversality Theorem.

Theorem A.1.9 (Relative Multijet Transversality Theorem)**.** *Let M, N be smooth manifolds, $X \subset J^k_s(M,N)$ a submanifold. Let furthermore $A \subset M$ be closed and $f_0 \colon M \to N$ be a smooth map such that $j^k_s f_0 \pitchfork X$ on an open neighbourhood U of A. Then there is a map g arbitrarily close to f_0 (in the C^∞-topology) such that $f_0|_A = g|_A$ and $j^k_s g \pitchfork X$.*

Sketch of proof. The nonrelative case is done in [GG73, Theorem 4.9 and Theorem 4.13], we will indicate which changes are needed for the relative version. We will further only sketch this proof for the case of $s = 1$, the changes for $s \geq 2$ are the same as in [GG73, Proof of Theorem 4.13]. First one chooses a countable set of open subsets $X_r \subset X$, $r \in \mathbb{N}$ such that

1. $\overline{X_r} \subset X \setminus \alpha^{-1}(A)$ and $\cup_{r \geq 0} X_r = X \setminus \alpha^{-1}(A)$.
2. $\alpha \times \beta(\overline{X_r}) \subset V_r \times V'_r$ for coordinate neighbourhoods $V_r \subset M, V'_r \subset N$ such that V_r is contained in $M \setminus A$.
3. $\overline{X_r}$ is compact.

For $B \subset X$ one defines

$$T_B := \{g \in C^\infty(M,N) \mid g \pitchfork X \text{ at } x \text{ for every } x \in M \text{ such that } j^s_k g(x) \in B \text{ and } g = f_0 \text{ on } A\}.$$

and shows that T_{X_r} is open and dense in $T_\emptyset$. This works along the same lines as in [GG73, Proof of Lemma 4.14]. Then

$$T := \bigcap_{r \geq 0} T_{X_r} = \left\{ g \in C^\infty(M,N) \colon \begin{array}{c} g \pitchfork X \text{ at } x \text{ for every } x \text{ with } j^s_k g(x) \in X_r \\ \text{and every } r \geq 0 \text{ and } g|_A = f_0|_A \end{array} \right\}$$

and hence every element of T is transverse to X and agrees with f_0 on A. Since $T_\emptyset$ is a Baire space T is dense. Therefore there is a map as requested in the Corollary. □

A.2 Applications

We first need to encode properties of maps into conditions on jets. Let $\Delta^n(X)$ denote the diagonal in $X^{\times n}$. We define

$$S_r(X,Y) := \{\sigma \in J^1(X,Y) \mid \operatorname{rank}(\sigma) = \dim Y - r\}.$$

This is a submanifold of the jet space for all $r \geq 0$.

Proposition A.2.1 ([GG73, Lemma 5.1 and Theorem 5.7])**.**
1. *$f\colon X \to Y$ is an immersion if and only if $j^1 f(X) \cap \big(\cup_{r\geq 1} S_r(X,Y)\big) = \emptyset$.*
2. *$f\colon X \to Y$ is injective if and only if $j^1_2 f(X) \cap (\beta^2)^{-1}\Delta^2(X) = \emptyset$.*

Proposition A.2.2. *An injective immersion from a compact manifold into a Hausdorff space is an embedding.*

Proof. This is clear, because in this case an injective map is a homeomorphism onto its image. □

Proposition A.2.3 ([GG73, Theorem 5.4])**.** *We have* $\operatorname{codim}(S_r(\mathbb{R}\times S^{k-1}, W)) = r(d-k+r+1)$ *and* $\operatorname{codim}(\Delta^2(W)) = d$.

Before we can give the proof of Lemma A.1 we need a few preparations.

Proof of Lemma A.1. We will show that both maps

$$\operatorname{Emb}(S^{k-1}\times D^{d-k+1}, V) \xrightarrow{(1)} \operatorname{Emb}(S^{k-1}\times D^{d-k+1}, W) \xrightarrow{(2)} \operatorname{Imm}(S^{k-1}\times D^{d-k+1}, W)$$

are π_0-bijections. Let us consider (1) first. Let $j \in \operatorname{Emb}(S^{k-1}\times D^{d-k+1}, W)$. Since (W,V) is $(k-1)$-connected, $j|_{S^{k-1}\times\{0\}}$ is homotopic to a map $f\colon S^{k-1} \hookrightarrow V$ which in turn is homotopic to an embedding $f'\colon S^{k-1} \hookrightarrow V$ by the Whitney embedding theorem. We need to turn this path into a path of embeddings. For this we define

$$A_r := \{\sigma \in J^1(\mathbb{R}\times S^{k-1}, W)\colon \sigma_t \in S_r(S^{k-1}, W)\}.$$

Then $\operatorname{codim} A_r = r\cdot(d-(k-1)+r)$ and if $d \geq 2k-1$, we have $\operatorname{codim} A_r > k$ for all $r \geq 1$. If $F\colon [0,1]\times S^{k-1} \to W$ is a path such that $j^1F \pitchfork A_r$ it also satisfies

$j^1F(\mathbb{R} \times S^{k-1}) \cap A_r = \emptyset$ and hence is a path of immersions. If furthermore $j_2^0 F((\mathbb{R} \times S^{k-1})^{(2)}) \cap (\beta^2)^{-1}(\Delta^2(W)) = \emptyset$, then F is a path of embeddings such that $F(t, \_)$ and $F(t', \_)$ have disjoint images for $t \neq t'$. For $d \geq 2k+1$ we have $\operatorname{codim} \Delta^2(W) > 2k = 2 \cdot \dim(\mathbb{R} \times S^{k-1})$. It follows from multijet transversality that the set of such paths F is residual. If $j|_{S^{k-1}\times\{0\}}$ and f' have disjoint images (which we can arranged by isotoping f'), then by relative multijet transversality there is a path of embeddings connecting them. By the isotopy extension theorem this can be extended to an isotopy of j and the map (1) is π_0-surjective. For π_0-injectivity let j, j' be embeddings into V that are isotopic through embeddings into V. Since (W, V) is $(k-1)$-connected this path can be homotoped into V and we can use the same argument as above.

Let us now consider map (2) from above. We define the following intermediate space

$$\operatorname{Imm}^0(S^{k-1} \times D^{d-k+1}, W) \coloneqq \{f \in \operatorname{Imm}(S^{k-1} \times D^{d-k+1}, W) \colon f|_{S^{k-1}\times\{0\}} \text{ is injective}\}.$$

We use the criterion from Proposition B.2 to show that the inclusion

$$\operatorname{Emb}(S^{k-1} \times D^{d-k+1}, W) \hookrightarrow \operatorname{Imm}^0(S^{k-1} \times D^{d-k+1}, W)$$

is a weak equivalence. Let $G_0 \colon D^n \to \operatorname{Imm}(S^{k-1} \times D^{d-k+1}, W)$ such that $G_0(S^{n-1}) \subset \operatorname{Emb}(S^{k-1} \times D^{d-k+1}, W)$. Since the disk D^n is compact, there exists an $\varepsilon > 0$ such that $G_0(x)|_{S^{k-1}\times D^{d-k+1}(\varepsilon)}$ is also injective hence an embedding because the source is compact. A homotopy of G_0 into $\operatorname{Emb}(S^{k-1} \times D^{d-k+1}, W)$ is then given by $G_\lambda(x)(p, v) \coloneqq G_0(x)(p, (1-\lambda(1-\varepsilon))v)$ for $(p, v) \in S^{k-1} \times D^{d-k+1}$. Thus it suffices to show that $\operatorname{Imm}^0(S^{k-1} \times D^{d-k+1}, W) \to \operatorname{Imm}(S^{k-1} \times D^{d-k+1}, W)$ is a π_0-bijection. Let $j \in \operatorname{Imm}(S^{k-1} \times D^{d-k}, W)$. Then $j|_{S^{k-1}\times\{0\}}$ is arbitrarily close to an embedding by the Whitney embedding theorem and since immersions are open in the space of smooth maps there is regular homotopy of j such that $j|_{S^{k-1}\times\{0\}}$ is injective. So the map is π_0-surjective. For injectivity let $j, j' \in \operatorname{Imm}^0(S^{k-1} \times D^{d-k+1}, W)$ be regularly homotopic and let F denote such a homotopy. The path $F|_{S^{k-1}\times\{0\}}$ connecting $j|_{S^{k-1}\times\{0\}}$ and $j'|_{S^{k-1}\times\{0\}}$ is homotopic to an isotopy f by relative Multijet transversality with the same argument as above. We extend this isotopy to a path F' connecting j and j'. Because immersions form an open subspace and because f can be chosen arbitrarily close to $F|_{S^{k-1}\times\{0\}}$ we may assume that F' is a regular homotopy that is injective when restricted to $S^{k-1} \times \{0\}$. □

Remark A.2.4. Lemma A.1 should also be true for $d \geq 2k$, since the given proof constructs a path such that f_t and $f_{t'}$ even have disjoint images for $t \neq t'$ which is not required.

Let us now turn to Morse functions. It is well known that the space of generalized Morse functions is connected, so we only need to show that there exists a *generic* path. We define

$$
\begin{aligned}
S_r(X) &:= S_r(X, \mathbb{R}) \\
S^n(X) &:= (S_1(X)^{\times n}) \cap (\beta^n)^{-1}(\Delta^n(\mathbb{R})) \subset J^1_n(X, \mathbb{R}) \\
S_{1,q}(X) &:= \{\sigma \in J^2(X, \mathbb{R}) \mid \sigma \in S_1(X) \text{ and } \mathrm{rank}(\mathrm{d^2f_p}) = \dim X - q \text{ if } [f] = \sigma\} \\
\tilde{S}_{1,1}(X) &:= \{\sigma \in J^3(X, \mathbb{R}) \mid \sigma \in S_{1,1}(X) \text{ and } d^3 f \neq 0 \text{ if } [f] = \sigma\} \\
\overline{S}_{1,1}(X) &:= \{\sigma \in J^3(X, \mathbb{R}) \mid \sigma \in S_{1,1}(X) \text{ and } d^3 f = 0 \text{ if } [f] = \sigma\}.
\end{aligned}
$$

Proposition A.2.5. *Let h be a path of generalized Morse functions.*

1. $j^3 h_t \cap \tilde{S}_{1,1}(W) \neq \emptyset$ *and* $j^3 h_t \cap \overline{S}_{1,1}(W) = \emptyset \iff h_t$ *has a birth-death-singularity.*
2. $j^3 h_t(p) \pitchfork S_{1,1}(W) \iff p$ *is a birth-death-point which is generically unfolded by h.*
3. $j^1_2 h_t(W) \cap S^2(W) \neq \emptyset \iff h_t$ *has two critical points with the same value.*
4. $j^1_3 h_t(W) \cap S^3(W) \neq \emptyset \iff h_t$ *has three critical points with the same value.*

Proof. 2. is [Igu88, Proposition 2.4, p.307]. The rest of the proof works by simply deciphering the conditions on submanifolds. □

Proposition A.2.6. *All of the above are submanifolds of the respective jet spaces and for $q \geq 1$ we have:*

$$
\begin{aligned}
\mathrm{codim}\ S_{1,q}(W) &= d + \frac{q(q+1)}{2} \\
\mathrm{codim}\ \tilde{S}_{1,1}(W) &= d + 1 \\
\mathrm{codim}\ \overline{S}_{1,1}(W) &> d + 1. \\
\mathrm{codim}\ S^n(W) &= n(d+1) - 1.
\end{aligned}
$$

Proof. Let us consider $S_{1,q}(W)$ first. Locally we have (compare [GG73, Lemma 2.6 and Theorem 2.7])

$$\begin{aligned} J^2(W,\mathbb{R}) &\cong W \times \mathbb{R} \times \operatorname{Hom}(\mathbb{R}^d,\mathbb{R}) \times \operatorname{Sym}_d \\ S_{1,q}(W) &\cong W \times \mathbb{R} \times \quad 0 \quad \times \underbrace{\{A \in \operatorname{Sym}_d \mid \operatorname{rank(A)} = d-q\}}_{=:\operatorname{Sym}_d^q} \end{aligned}$$

So we need to compute the codimension of Sym_d^q in Sym_d. This is analogous to [GG73, Lemma 5.2]. If a matrix A has rank $d-q$, there is a basis of $\mathbb{R}^d$ such that $A = \begin{pmatrix} B & C \\ C^T & D \end{pmatrix}$ where B is an invertible $(d-q)\times(d-q)$-matrix and B and D are symmetric. Then there is a neighbourhood U of A such that the map

$$g\colon U \to \operatorname{Sym}_q, \quad g\begin{pmatrix} B & C \\ C^T & D \end{pmatrix} = D - C^T BC$$

has 0 as a regular value and $f^{-1}(0) = \operatorname{Sym}_d^q$ which is therefore a submanifold of Sym_d with $\operatorname{codim} \operatorname{Sym}_d^q = \dim \operatorname{Sym}_q = \frac{q(q+1)}{2}$.

For $\tilde{S}_{1,1}(N)$ we note that $S_{1,1}(N)$ can also be seen as a submanifold of $J^3(N,\mathbb{R})$ (as the preimage of the projection map $J^3(N,\mathbb{R}) \to J^2(N,\mathbb{R})$) and the additional condition on $\tilde{S}_{1,1}(N)$ is an open condition, so $\tilde{S}_{1,1}(N)$ is again a submanifold and its codimension is the same as the one of $S_{1,1}(N)$, namely $n+1$.

The estimate on $\overline{S}_{1,1}(W)$ follows from the fact that $d^3 f = 0$ means that the 3-jet of f is 0 and therefore this is a submanifold of $S_{1,1}(N)$ of strictly positive codimension.

For the last equality we note that

$$\begin{aligned} J^1_n(W,\mathbb{R}) &\cong W^{\times n} \times \mathbb{R}^n \times \operatorname{Hom}(\mathbb{R}^d,\mathbb{R})^n \\ S^n(W) &\cong W^{\times n} \times \Delta_n(\mathbb{R}) \times 0 \end{aligned}$$

and so $\operatorname{codim} S^n(W) = n - 1 + nd$. □

Proof of Lemma A.2. The proof will now work by encoding the desired properties of the path h_t into conditions on the jets of h_t. We define

$$\begin{aligned} A &:= \{\sigma \in J^3(\mathbb{R} \times W, \mathbb{R}\colon \sigma_t \in S_{1,1}\} \\ \tilde{A} &:= \{\sigma \in J^3(\mathbb{R} \times W, \mathbb{R}\colon \sigma_t \in \tilde{S}_{1,1}\} \\ \overline{A} &:= \{\sigma \in J^3(\mathbb{R} \times W, \mathbb{R}\colon \sigma_t \in \overline{S}_{1,1}\} \\ B^n &:= \{\sigma \in J^1_n(\mathbb{R} \times W, \mathbb{R}\colon \sigma_t \in S^n\} \end{aligned}$$

These submanifolds have the same codimensions as $S_{1,1}, \tilde{S}_{1,1}, \overline{S}_{1,1}$ and S^n respectively. Let $h \in C^\infty(\mathbb{R} \times W, \mathbb{R})$. If $j^3 h \pitchfork A$, then by Proposition A.2.6 $j^3 h(\mathbb{R} \times W) \cap A$ is a finite collection of points. The same holds for $\tilde{A}$. If $j^3 h \pitchfork \overline{A}$ the intersection is empty. Let $C := (\alpha^2)^{-1}(\Delta(\mathbb{R}) \times W^{(2)}) \cap A^2$. If $j^3_2((t,p),(t',p')) \in C$, then $t = t'$ and $p \neq p'$ are both nondegenerate singularities of h_t. $C \subset J^3_2(\mathbb{R} \times W, \mathbb{R})$ is a submanifold of codimension $2d+3$ and hence if $j^3_2 h \pitchfork C$, then $j^3_2(W) \cap C = \emptyset$. The set of functions satisfying these four conditions therefore are paths of generalized Morse functions with only finitely many, generically unfolded birth-death-singularities which appear at different times.

Let $\tilde{B}^n := \alpha_n^{-1}(\Delta_n(\mathbb{R}) \cap B^n$ where $\Delta_n(\mathbb{R})$ denotes the 1-dimensional diagonal in $\mathbb{R}^n$. If $j^1_2 h \pitchfork \tilde{B}^2$, then $j^1_2 h(\mathbb{R} \times W) \cap \tilde{B}^2 \subset J^1_2(\mathbb{R} \times W, \mathbb{R})$ is a submanifold of codimension $2d+2$ and hence so is $(j^1_2 h)^{-1}(\tilde{B}^2) \subset (\mathbb{R} \times W)^2$ which therefore is a finite collection of points. So, if $j^1_2 h \pitchfork \tilde{B}^2$ there are only finitely many times where h_t has non-distinct critical values. If there exists a t such that h_t has three critical points p, p', p'' with the same value, then $j^1_3 h((\mathbb{R} \times W)^3) \cap \tilde{B}^3 \neq \emptyset$. But $\tilde{B}^3$ is a submanifold of codimension $3d+4$ and if $j^1_3 h \pitchfork \tilde{B}^3$ the before-mentioned intersection is empty. If there is a t such that h_t has 2 critical values with 2 preimages each, then $j^1_4 h \cap \alpha_4^{-1}(\Delta_4(\mathbb{R})) \cap (S^2(W) \times S^2(W)) \neq \emptyset$. The submanifold $\alpha_4^{-1}(\Delta_4(\mathbb{R})) \cap (S^2(W) \times S^2(W))$ has codimension $3+2(2d+1) = 4d+5$ and again, if $j^1_4 h \pitchfork \alpha_4^{-1}(\Delta_4(\mathbb{R})) \cap (S^2(W) \times S^2(W))$, the corresponding intersection is empty.

The final property to encode is that if h_t has a birth-death-point, then critical values of h_t are distinct. Let h_t have a birth-death-point and two critical points with the same value. Then $j^3_3 h \cap \alpha_3^{-1}(\Delta_3(\mathbb{R})) \cap (S_{1,1} \times S^2)$ would be nonempty. But $\alpha_3^{-1}(\Delta_3(\mathbb{R})) \cap (S_{1,1} \times S^2)$ is a submanifold of codimension $2 + d + 1 + 2d + 1 = 3d + 4$ and the same argument as above applies.

So, let $h\colon \mathbb{R} \times W \to \mathbb{R}$ be a function that is constantly equal to h_0 near $\{0\} \times W$ and equal to h_1 near $\{1\} \times W$ which satisfies:

1. j^3h is transverse to $A, \tilde{A}, \overline{A}$ and j^3_2h is transverse to C
2. j^1_2h is transverse to D^2
3. j^1_3h is transverse to D^3 and j^1_4h is transverse to $\alpha_4^{-1}(\Delta_4(\mathbb{R})) \cap (S^2 \times S^2)$
4. j^3_3 is transverse to $\alpha_3^{-1}(\Delta_3(\mathbb{R})) \cap (S_{1,1} \times S^2)$.

By the relative relative Multijet-Transversality Theorem A.1.9 the set of such functions h is residual and hence nonempty. □

B

Miscellaneous

Lemma B.1. *Let $\pi\colon E \to B$ be a Serre-fibration and let X be a CW-complex. Then $\mathrm{Map}(X, E) \to \mathrm{Map}(X, B)$ is a Serre-fibration.*

Proof. Consider the following lifting problem for D a CW-complex:

$$\begin{array}{ccc}
\{0\} \times D & \longrightarrow & \mathrm{Map}(X, E) \\
\big\downarrow & \overset{F}{\nearrow} & \big\downarrow \\
[0, 1] \times D & \xrightarrow{\ f\ } & \mathrm{Map}(X, B).
\end{array}$$

We obtain the following diagram

$$\begin{array}{ccc}
\{0\} \times D \times X & \longrightarrow & E \\
\big\downarrow & \overset{\hat{F}}{\nearrow} & \big\downarrow \\
[0, 1] \times D \times X & \xrightarrow{\ \hat{f}\ } & B
\end{array}$$

by $\hat{f}(t,d,x) := f(t,d)(x)$. Since $E \to B$ is a fibration and D is a CW-complex, the map $\hat{F}$ exists and we define $F(t,d)(x) := \hat{F}(t,d,x)$. □

Proposition B.2. *Let $j : X \to Y$ be the inclusion of a subspace. Then the following are equivalent:*

1. *j is a weak homotopy equivalence,*
2. *for every $n \geq 0$ and every map $G_0\colon D^n \to Y$ such that $G(S^{n-1}) \subset X$, there exists a homotopy G_λ starting with G_0 such that $G_1(D^n) \subset X$ and $G_\lambda(S^{n-1}) \subset X$ for all $\lambda \in [0,1]$.*

Proof. (2) ⇒ (1): We show that G_0 is homotopic relative to S^{n-1} to a map G'_1 into X and invoke a standard lemma (e.g. [Gra75, p. 136]). Let $G_0 : D^n \to Y$ be a map with $G_0(S^{n-1}) \subset X$. The map G_0 is homotopic, relative to S^{n-1}, to the map

$$G'_{\frac{1}{2}}(x) := \begin{cases} G_0(\frac{x}{\|x\|}) & \|x\| \geq \frac{1}{2}, \\ G_0(2x) & \|x\| \leq \frac{1}{2}. \end{cases}$$

For $\lambda \geq \frac{1}{2}$, define

$$G'_\lambda(x) := \begin{cases} G_{(2-2\|x\|)(2\lambda-1)}\left(\frac{x}{\|x\|}\right) & \|x\| \geq \frac{1}{2}, \\ G_{2\lambda-1}(2x) & \|x\| \leq \frac{1}{2}. \end{cases}$$

The other implication is likewise easy and not important for us. □

Lemma B.3. *Let $M^m \subset N^n$ be a compact submanifold of codimension $r = n - m$. Then the inclusion $N \backslash \tau(M) \hookrightarrow N$ is $r-1$-connected for $\tau(M)$ any small enough tubular neighbourhood of M.*

Proof. Let $k \leq r-1$ and $f\colon (D^k, S^{k-1}) \to (N, N \setminus M)$ be a map. This is homotopic to a smooth map f_s by the theorem of Stone-Weierstraß which in turn is homotopic to $f_\pitchfork$ which is transverse to M. Hence $\operatorname{im} f \cap M$ is a submanifold of dimension $k - r < 0$ and hence $0 = [f] \in \pi_k(N, N \setminus M)$. The tubular neighbourhood statement follows from compactness of M and the fact that $f_\pitchfork$ has distance greater ε from M for some $\varepsilon > 0$. □

Lemma B.4 ([Kre99, Proposition 4], [HJ13, Proposition, Appendix III])**.** *Let $\theta\colon B \to BO(m)$ be a tangential structure, with B of type F_n. Let $W^m\colon M_0 \rightsquigarrow M_1$ be a θ-cobordism* *and let $M_1 \to B$ be n-connected. If $n \leq \frac{m}{2} - 1$, there exists a θ-cobordism $W'\colon M_0 \rightsquigarrow M_1$ such that (W', M_1) is n-connected. If furthermore $M_0 \to B$ is also n-connected, there exists a θ-cobordism $W'\colon M_0 \rightsquigarrow M_1$ such that (W', M_i) is n-connected for $i = 0, 1$. Furthermore W' is θ-cobordant to W relative to the boundary.*

Proof. We may perform surgery on the interior of W to turn $W \to B$ into an n-connected. From the long exact sequence for the triple (B, W, M_i) we get that $M_i \hookrightarrow W$ is an isomorphism on π_k for $k \leq n-1$. It remains to show that it can be made surjective on π_n. Consider the sequence

$$\begin{array}{ccccccccc} \pi_n(M_0) & \xrightarrow{\;c\;} & \pi_n(W) & \xrightarrow{\;b\;} & \pi_n(W, M_0) & \xrightarrow{\;0\;} & \pi_{n-1}(M_0) & \xrightarrow{\;\cong\;} & \pi_{n-1}(W) \\ & \searrow & \downarrow{\scriptstyle a} & & & & & & \\ & & \pi_n(B) & & & & & & \end{array}$$

Since $\pi_n(W, M_0)$ is a finitely generated $\mathbb{Z}\pi_1$-module and b is surjective, we find elements $x_1, \ldots, x_l \in \pi_n(W)$ that are mapped to generators. Also, there are preimages $y_1, \ldots, y_l \in \pi_n(M_0)$ of $a(x_1), \ldots, a(x_l)$. Let $z_i := x_i - c(y_i)$. Then the $b(z_i)$ still are generators and $a(z_i) = 0$. For this reason and by the Whitney embedding theorem one can assume that z_i are represented by embeddings φ_i with trivial normal bundle and hence they can be surgered away. Therefore we get a cobordism W' such that (W', M_0) is n-connected and W' is obtained by performing $n+1$-surgeries on the interior of W. Furthermore, $a \circ \varphi_i|_{S^n \times \{0\}}$ is nullhomotopic and hence it can be extended to a map $\tilde{\varphi}_i\colon D^{n+1} \to B$ and we get a structure map $W_{\varphi_i} \to B$. So, W' is also a θ-cobordism.

We now do the same trick for M_1 but we have to show that the connectivity of (W, M_0) is not destroyed if an $n+1$-surgery is performed. Let $\varphi\colon S^n \times D^{d-n} \hookrightarrow W$ be an $n+1$-surgery embedding and let $\mathring{W} := W \setminus \operatorname{im} \varphi$. We get

$$\begin{array}{ccccc} W & \xleftarrow{\;d-n-1\text{-connected}\;} & \mathring{W} & \xrightarrow{\;n\text{-connected}\;} & W_\varphi \\ & {\scriptstyle n\text{-connected}}\nwarrow & \uparrow & \nearrow & \\ & & M_0 & & \end{array}$$

Since $d - n - 1 \geq \frac{d}{2} > n$, the middle vertical map and hence $M_0 \hookrightarrow W_\varphi$ are n-connected. ☐

Lemma B.5. *Let $f\colon X \to X$ be a map. Then there exists a long exact sequence*

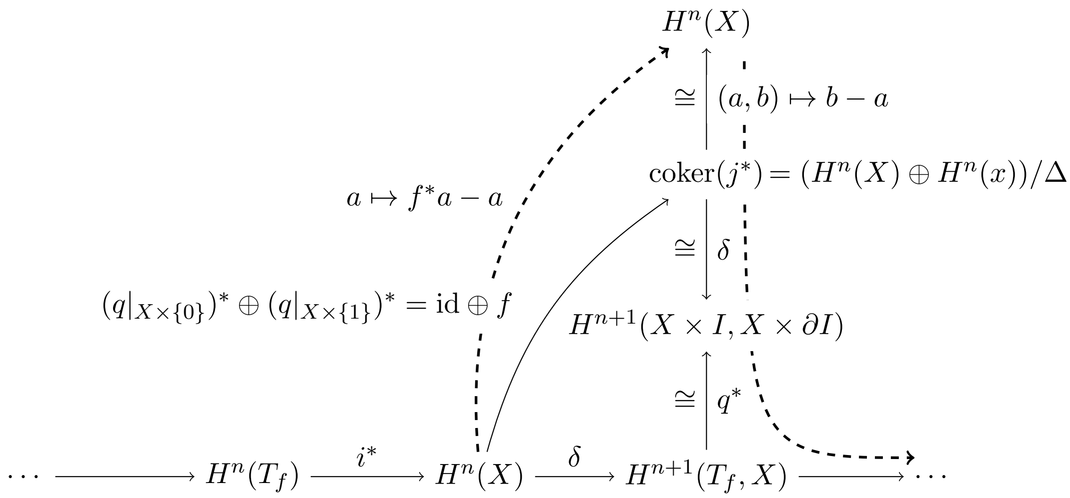
$$\cdots \to H^n(X) \xrightarrow{f^*-1} H^n(X) \xrightarrow{\delta} H^{n+1}(T_f) \xrightarrow{i^*} H^{n+1}(X) \to \ldots$$

which is called the Wang sequence. This reduces to a short exact sequence

$$0 \longrightarrow H^n(X)_f \xrightarrow{\delta} H^{n+1}(T_f) \xrightarrow{i^*} H^{n+1}(X)^f \longrightarrow 0$$

where A_f denotes the coinvariants and A^f the invariants of A with respect to f.

Proof. This is dual to [Hat02, Example 2.48]. Consider the quotient map $q\colon (X \times I, X \times \partial I) \to (T_f, X)$. We have the following diagram:



where j^* and the vertical map δ come from the long exact sequence for $(X \times I, X \times \partial I)$. The dashed arrows give the desired sequence. ☐

Bibliography

[AB02] K. Akutagawa and B. Botvinnik. The relative Yamabe invariant. *Comm. Anal. Geom.*, 10(5):935–969, 2002.

[ABP67] D. W. Anderson, E. H. Brown, Jr., and F. P. Peterson. The structure of the Spin cobordism ring. *Ann. of Math. (2)*, 86:271–298, 1967.

[Ada93] M. Adachi. *Embeddings and Immersions*. Translations of mathematical monographs. American Mathematical Society, 1993.

[AS63] M. F. Atiyah and I. M. Singer. The index of elliptic operators on compact manifolds. *Bull. Amer. Math. Soc.*, 69:422–433, 1963.

[BERW17] B. Botvinnik, J. Ebert, and O. Randal-Williams. Infinite loop spaces and positive scalar curvature. *Invent. Math.*, 209(3):749–835, 2017.

[BHS64] H. Bass, A. Heller, and R. G. Swan. The Whitehead group of a polynomial extension. *Inst. Hautes Études Sci. Publ. Math.*, (22):61–79, 1964.

[BHSW10] B. Botvinnik, B. Hanke, T. Schick, and M. Walsh. Homotopy groups of the moduli space of metrics of positive scalar curvature. *Geom. Topol.*, 14(4):2047–2076, 2010.

[Bre93] G.E. Bredon. *Topology and Geometry*. Graduate texts in mathematics. Springer, 1993.

[Car88] R. Carr. Construction of manifolds of positive scalar curvature. *Trans. Amer. Math. Soc.*, 307(1):63–74, 1988.

[Cer70] J. Cerf. La stratification naturelle des espaces de fonctions différentiables réelles et le théorème de la pseudo-isotopie. *Inst. Hautes Études Sci. Publ. Math.*, 39:5–173, 1970.

[Che04a] V. Chernysh. A quasifibration of spaces of positive scalar curvature metrics. *ArXiv Mathematics e-prints*, May 2004.

[Che04b] V. Chernysh. On the homotopy type of the space $\mathcal{R}^+(M)$. *ArXiv Mathematics e-prints*, May 2004.

[CS13] D. Crowley and T. Schick. The Gromoll filtration, KO-characteristic classes and metrics of positive scalar curvature. *Geom. Topol.*, 17(3):1773–1789, 2013.

[CSS16] D. Crowley, T. Schick, and W. Steimle. Harmonic spinors and metrics of positive curvature via the Gromoll filtration and Toda brackets. *ArXiv e-prints*, December 2016.

[Die08] T. Dieck. *Algebraic Topology*. EMS textbooks in mathematics. European Mathematical Society, 2008.

[Ebe06] J. Ebert. *Characteristic classes of spin surface bundles: applications of the Madsen-Weiss theory*, volume 381 of *Bonner Mathematische Schriften [Bonn Mathematical Publications]*. Universität Bonn, Mathematisches Institut, Bonn, 2006. Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, 2006.

[EF18] J. Ebert and G. Frenck. The Gromov-Lawson-Chernysh surgery theorem. *ArXiv e-prints*, July 2018.

[ERW17a] J. Ebert and O. Randal-Williams. Infinite loop spaces and positive scalar curvature in the presence of a fundamental group. *ArXiv e-prints*, November 2017.

[ERW17b] J. Ebert and O. Randal-Williams. Semi-simplicial spaces. *ArXiv e-prints*, May 2017.

[GG73] M. Golubitsky and V. Guillemin. *Stable mappings and their singularities*. Springer-Verlag, New York-Heidelberg, 1973. Graduate Texts in Mathematics, Vol. 14.

[GL80] M. Gromov and H. B. Lawson, Jr. The classification of simply connected manifolds of positive scalar curvature. *Ann. of Math. (2)*, 111(3):423–434, 1980.

[GL83] M. Gromov and H. B. Lawson, Jr. Positive scalar curvature and the Dirac operator on complete Riemannian manifolds. *Inst. Hautes Études Sci. Publ. Math.*, pages 83–196 (1984), 1983.

[Gra75] B. Gray. *Homotopy Theory: An Introduction to Algebraic Topology*. Pure and Applied Mathematics. Elsevier Science, 1975.

[GRW10] S. Galatius and O. Randal-Williams. Monoids of moduli spaces of manifolds. *Geom. Topol.*, 14(3):1243–1302, 2010.

[GRW14] S. Galatius and O. Randal-Williams. Stable moduli spaces of high-dimensional manifolds. *Acta Math.*, 212(2):257–377, 2014.

[GRW16] S. Galatius and O. Randal-Williams. Abelian quotients of mapping class groups of highly connected manifolds. *Math. Ann.*, 365(1-2):857–879, 2016.

[Haj88] B. Hajduk. Metrics of positive scalar curvature on spheres and the Gromov-Lawson conjecture. *Math. Ann.*, 280(3):409–415, 1988.

[Hat75] A. E. Hatcher. Higher simple homotopy theory. *Ann. of Math. (2)*, 102(1):101–137, 1975.

[Hat02] A. Hatcher. *Algebraic Topology*. Algebraic Topology. Cambridge University Press, 2002.

[Hat17] A Hatcher. Vector bundles & k-theory. incomplete book draft, version 2.2, 11 2017.

[Hir76] M.W. Hirsch. *Differential Topology*. Graduate Texts in Mathematics. Springer New York, 1976.

[Hit74] N. Hitchin. Harmonic spinors. *Advances in Math.*, 14:1–55, 1974.

[HJ13] F. Hebestreit and M. Joachim. Twisted Spin cobordism and positive scalar curvature. *ArXiv e-prints*, November 2013.

[Hoe16] S. Hoelzel. Surgery stable curvature conditions. *Math. Ann.*, 365(1-2):13–47, 2016.

[HSS14] B. Hanke, T. Schick, and W. Steimle. The space of metrics of positive scalar curvature. *Publ. Math. Inst. Hautes Études Sci.*, 120:335–367, 2014.

[Igu88] K. Igusa. The stability theorem for smooth pseudoisotopies. *K-Theory*, 2(1-2):vi+355, 1988.

[KL05] M. Kreck and W. Lück. *The Novikov conjecture*, volume 33 of *Oberwolfach Seminars*. Birkhäuser Verlag, Basel, 2005. Geometry and algebra.

[Kor18] J.-B. Kordaß. On the space of riemannian metrics satisfying surgery stable curvature conditions. *ArXiv e-prints*, August 2018.

[Kra] A. Krause. Is $su(3)/so(3)$ cobordant with a mapping torus? MathOverflow. URL:https://mathoverflow.net/q/164720 (version: 2014-04-30).

[Kre76] M. Kreck. Cobordism of odd-dimensional diffeomorphisms. *Topology*, 15(4):353–361, 1976.

[Kre99] M. Kreck. Surgery and duality. *Ann. of Math. (2)*, 149(3):707–754, 1999.

[Lab97] M.-L. Labbi. Stability of the p-curvature positivity under surgeries and manifolds with positive Einstein tensor. *Ann. Global Anal. Geom.*, 15(4):299–312, 1997.

[Lic63] A. Lichnerowicz. Spineurs harmoniques. *C. R. Acad. Sci. Paris*, 257:7–9, 1963.

[Mac71] S. MacLane. *Categories for the working mathematician*. Springer-Verlag, New York-Berlin, 1971. Graduate Texts in Mathematics, Vol. 5.

[Mat02] Y. Matsumoto. *An introduction to Morse theory*, volume 208 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 2002. Translated from the 1997 Japanese original by Kiki Hudson and Masahico Saito, Iwanami Series in Modern Mathematics.

[Mil65] J. W. Milnor. *Lectures on the h-cobordism theorem*. Notes by L. Siebenmann and J. Sondow. Princeton University Press, Princeton, N.J., 1965.

[MS74] J. W. Milnor and J. D. Stasheff. *Characteristic classes*. Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974. Annals of Mathematics Studies, No. 76.

[MT91] M. Mimura and H. Toda. *Topology of Lie Groups, I and II*. Translations of Mathematical Monographs. American Mathematical Society, 1991.

[MW07] I. Madsen and M. Weiss. The stable moduli space of Riemann surfaces: Mumford's conjecture. *Ann. of Math. (2)*, 165(3):843–941, 2007.

[Neu71] W. D. Neumann. Fibering over the circle within a bordism class. *Math. Ann.*, 192:191–192, 1971.

[Pal66] R. S. Palais. Homotopy theory of infinite dimensional manifolds. *Topology*, 5:1–16, 1966.

[Per17] N. Perlmutter. Cobordism Categories and Parametrized Morse Theory. *ArXiv e-prints*, March 2017.

[Sch98] T. Schick. A counterexample to the (unstable) Gromov-Lawson-Rosenberg conjecture. *Topology*, 37(6):1165–1168, 1998.

[Sma62] S. Smale. On the structure of manifolds. *Amer. J. Math.*, 84:387–399, 1962.

[Sto92] S. Stolz. Simply connected manifolds of positive scalar curvature. *Ann. of Math. (2)*, 136(3):511–540, 1992.

[SY79] R. Schoen and S. T. Yau. On the structure of manifolds with positive scalar curvature. *Manuscripta Math.*, 28(1-3):159–183, 1979.

[Tho54] R. Thom. Quelques propriétés globales des variétés différentiables. *Comment. Math. Helv.*, 28:17–86, 1954.

[TW15] W. Tuschmann and D. J. Wraith. *Moduli spaces of Riemannian metrics*, volume 46 of *Oberwolfach Seminars*. Birkhäuser Verlag, Basel, 2015. Second corrected printing.

[Wal60] C. T. C. Wall. Determination of the cobordism ring. *Ann. of Math. (2)*, 72:292–311, 1960.

[Wal71] C. T. C. Wall. Geometrical connectivity. I. *J. London Math. Soc. (2)*, 3:597–604, 1971.

[Wal11] M. Walsh. Metrics of positive scalar curvature and generalised Morse functions, Part I. *Mem. Amer. Math. Soc.*, 209(983):xviii+80, 2011.

[Wal13] M. Walsh. Cobordism invariance of the homotopy type of the space of positive scalar curvature metrics. *Proc. Amer. Math. Soc.*, 141(7):2475–2484, 2013.

[Wal14] M. Walsh. Metrics of positive scalar curvature and generalised Morse functions, Part II. *Trans. Amer. Math. Soc.*, 366(1):1–50, 2014.

[Wal16] C. T. C. Wall. *Differential topology*. Cambridge University Press, 2016.

[Win71] H. E. Winkelnkemper. *ON EQUATORS OF MANIFOLDS AND THE ACTION OF Θ('N)*. ProQuest LLC, Ann Arbor, MI, 1971. Thesis (Ph.D.)–Princeton University.

[Wol12] J. Wolfson. Manifolds with k-positive Ricci curvature. In *Variational problems in differential geometry*, volume 394 of *London Math. Soc. Lecture Note Ser.*, pages 182–201. Cambridge Univ. Press, Cambridge, 2012.

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