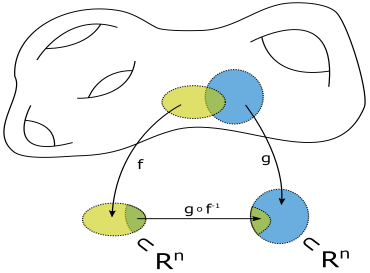
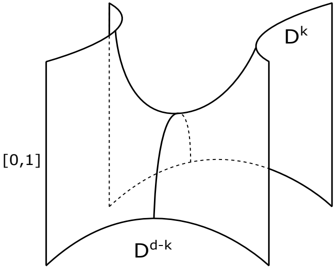
Karlsruher Institut für Technologie

The action of $\mathrm{Diff}(M)$ on $\mathcal{R}^+(M)$

Georg Frenck | November 27, 2019

INSTITUTE OF ALGEBRA AND GEOMETRY (IAG)





Introduction and Motivation

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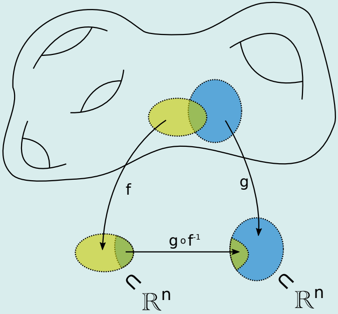

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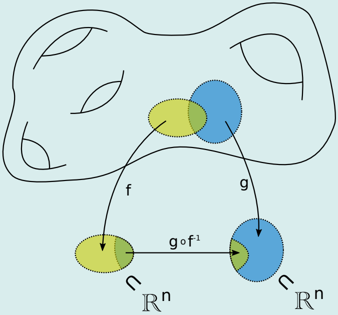

A manifold is called **smooth** if the transition functions are smooth.

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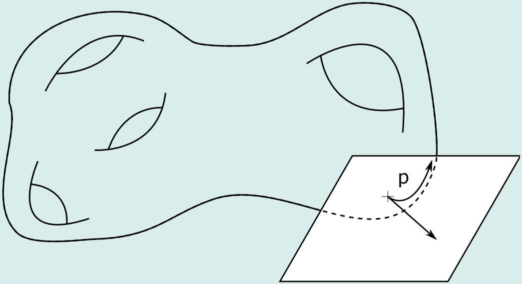

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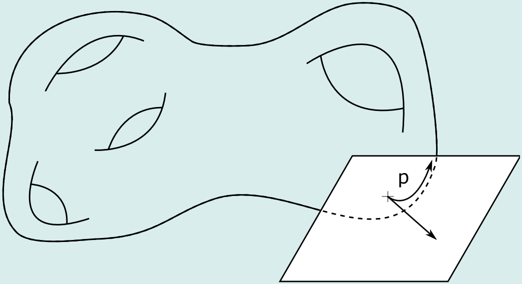

The collection of all tangent spaces is called the **tangent bundle** TM.

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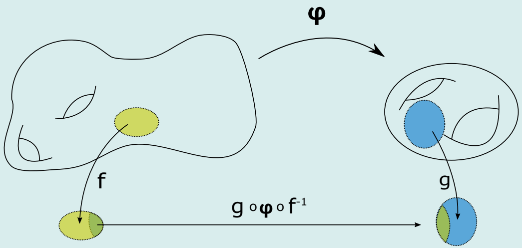

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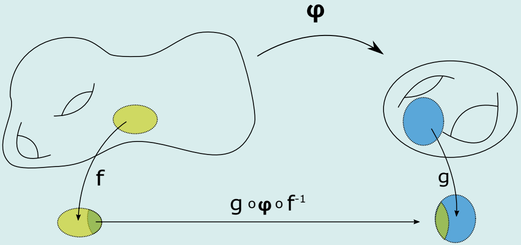

We denote the **group of self-diffeomorphisms** of M by $\mathrm{Diff}(M)$.

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Definition

A diffeomorphism $\varphi\colon (M, g) \to (M', g')$ of Riemannian manifolds is called **isometry** if $\varphi^* g' = g$.

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$$\mathcal{R}(M) \text{ looks like a point} \iff \mathcal{R}^+(M) \text{ is convex}$$
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$\rightsquigarrow$ Find isometry invariants of Riemannian metrics!

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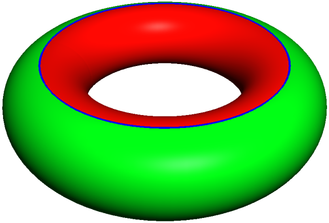
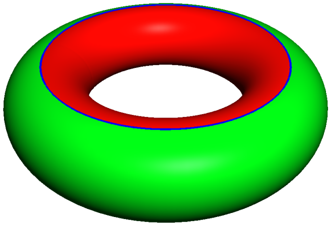


Scalar curvature

Riemannian metric $\longrightarrow$ Measure length and angles $\longrightarrow$ Measure volume

$\rightsquigarrow$ Compare (infinitesimal) volume to euclidian volume



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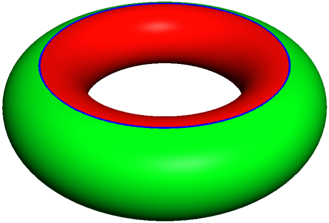

(M, g) Riem. manifold $\rightsquigarrow$ Scalar curvature: $\mathbf{scal}_g : M \longrightarrow \mathbb{R}$

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  - *S^n admits a metric of positive scalar curvature.*
  - *$(S^1)^n$ does not.*

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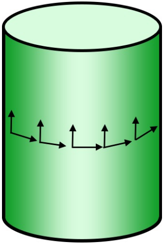

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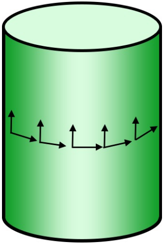

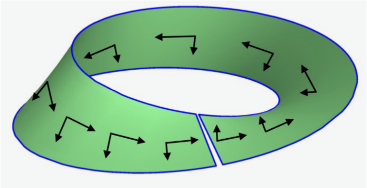

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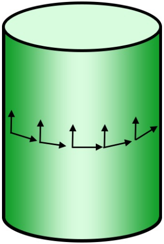

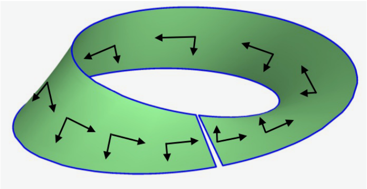

Definition

Let M be an oriented manifold. A Spin-**structure** on M is a "higher orientation" of M.

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The space $\mathcal{R}^+(M)$

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$\rightsquigarrow$ Gromov–Lawson–Rosenberg conjecture

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Let M be a Spin-manifold of dimension $(d-1)$. Then there is a map $\pi_k(\mathcal{R}^+(M)) \to \mathrm{KO}^{-d-k}(\mathrm{pt})$...

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- Construct a family $(g_s)_{s\in S^k} \subset \mathcal{R}^+(M)$.
- Show that this family gets mapped to a nontrivial element.

The space $\mathcal{R}^+(M)$

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- Consider the action $\mathrm{Diff}(M) \curvearrowright \mathcal{R}^+(M)$.

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Theorem (Hitchin 1974)

Let M be a Spin-manifold of dimension $(d-1)$. Then there is a map $\pi_k(\mathcal{R}^+(M)) \to \mathrm{KO}^{-d-k}(\mathrm{pt})$, such that

$$\begin{aligned} \pi_0(\mathrm{Diff}(S^{d-1})) \to \pi_0(\mathcal{R}^+(S^{d-1})) &\to \mathrm{KO}^{-d}(\mathrm{pt}) \\ [f] &\mapsto [f^* g_\circ] \end{aligned}$$

is surjective for $d \equiv 1, 2(8)$.

The space $\mathcal{R}^+(M)$

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Theorem (Hanke-Schick-Steimle, 2012)

If m is big enough, there exists a simply connected Spin-manifold M^{4m-1} such that

$$\pi_0(\mathrm{Diff}(M)) \to \pi_0(\mathcal{R}^+(M)) \to \mathrm{KO}^{-4m}(\mathrm{pt}) \cong \mathbb{Z}$$

is nontrivial.

The space $\mathcal{R}^+(M)$

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Question

Does this work for $M = S^{4m-1}$?

The space $\mathcal{R}^+(M)$

Theorem (F.)

Let $d \geq 7$, $d \not\equiv 1, 2(8)$ and let M^{d-1} be as simply connected, stably parallelizable Spin-manifold. Let $f\colon M \to M$ be an orientation preserving diffeomorphism. Then $f^*\colon \mathcal{R}^+(M) \to \mathcal{R}^+(M)$ is homotopic to the identity.

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1. $f^* \sim \mathrm{id}$

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Remark

1. $f^* \sim \mathrm{id} \Rightarrow \pi_0(f^*) = id\colon \pi_0(\mathcal{R}^+(M)) \to \pi_0(\mathcal{R}^+(M))$.

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Let $d \geq 7$, $d \not\equiv 1, 2(8)$ and let M^{d-1} be as simply connected, stably parallelizable Spin-manifold. Let $f\colon M \to M$ be an orientation preserving diffeomorphism. Then $f^*\colon \mathcal{R}^+(M) \to \mathcal{R}^+(M)$ is homotopic to the identity.

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1. $f^* \sim \mathrm{id} \Rightarrow \pi_0(f^*) = id\colon \pi_0(\mathcal{R}^+(M)) \to \pi_0(\mathcal{R}^+(M))$.
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Karlsruher Institut für Technologie

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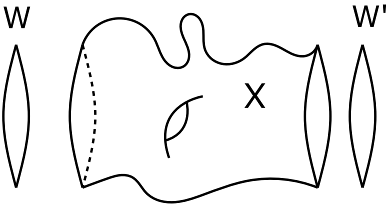

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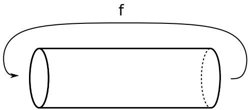

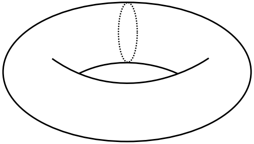

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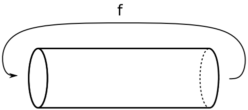

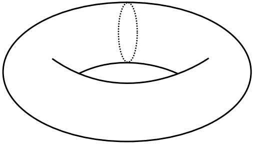

Proposition (1973 Kreck)

If M satisfies the assumptions from before, then $[T_f] = 0 \in \Omega_d^{\mathrm{Spin}}$ for every $f\colon M \xrightarrow{\cong} M$.

Construction of $\mathcal{S}$

Simplest piece: **tr** $\varphi\colon M \rightsquigarrow M_\varphi$ for $\varphi\colon S^{k-1} \times D^{d-k} \hookrightarrow M$

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$\iota\colon \mathcal{R}^+(M,\varphi)\hookrightarrow\mathcal{R}^+(M)$ *is a homotopy equivalence for* $d-k\geq 3$.

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$$V = \textbf{tr}\,\varphi_1 \cup \cdots \cup \textbf{tr}\,\varphi_n$$

such that all surgeries φ_i satisfy $d - k \geq 3$.

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Question

Is (the homotopy class of) $\mathcal{S}_W$ independent of these choices?

Construction of $\mathcal{S}$

Theorem ( )

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If $d \geq 7$, the (homotopy class of the) map $\mathcal{S}_W$ is independent of the choice of handle decomposition of V.

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Thank you for your attention!