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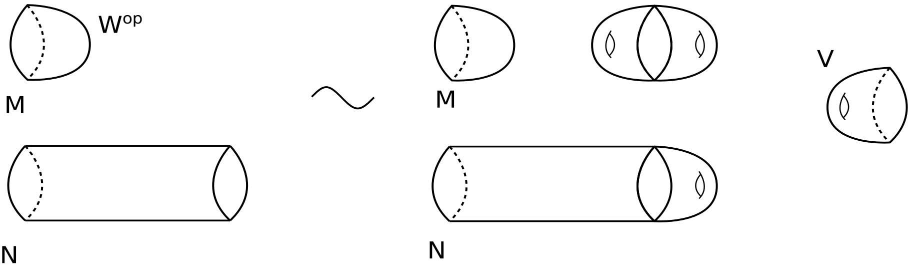
H-space structures on $\mathcal{R}^+(M)$

Georg Frenck | November 27, 2019

INSTITUTE OF ALGEBRA AND GEOMETRY (IAG)



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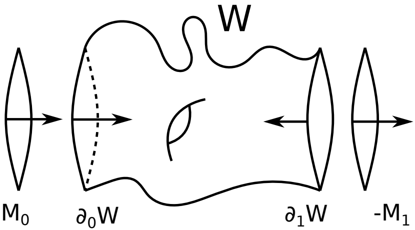

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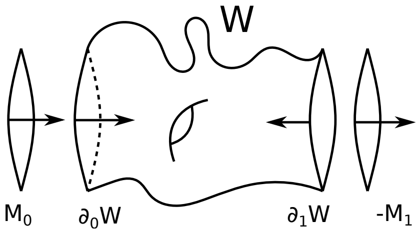

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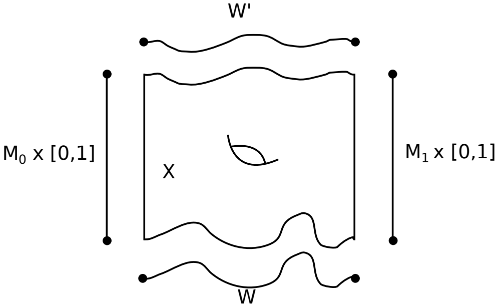

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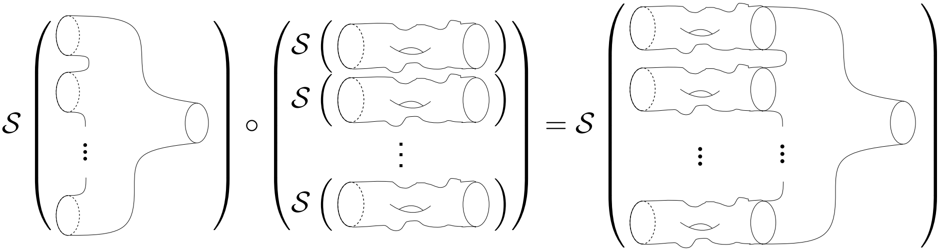

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An *H-space* is a tripel (X, μ, e) where $e \in X$ and μ is the homotopy class of a map $X \times X \to X$ such that

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An equivalence of H-spaces $(X_0, \mu_0, e_0), (X_1, \mu_1, e_1)$ is a (weak) homotopy equivalence $\varphi\colon X_0 \to X_1$ such that

- $\varphi \circ \mu_0$ is homotopic to $\mu_1 \circ (\varphi, \varphi)$
- e_1 lies in the same component as $\varphi(e_0)$.

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3. e is unique up to homotopy $\rightsquigarrow$ suffices to specify its path component.

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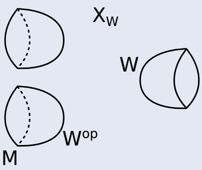

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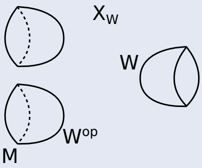

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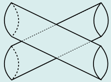
$\mu_W = \mathcal{S}(X_W), \tau = \mathcal{S}(T)$ for T = 

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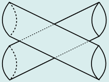
$\mu_W = \mathcal{S}(X_W)$, $\tau = \mathcal{S}(T)$ for T = 

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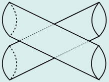

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$\rightsquigarrow$ suffices to show: $T \cup X_W \sim X_W$

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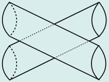
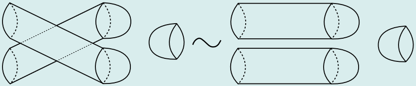
$\mu_W = \mathcal{S}(X_W)$, $\tau = \mathcal{S}(T)$ for T = 

$$\Rightarrow \mu_W \circ \tau = \mathcal{S}(X_W) \circ \mathcal{S}(T) = \mathcal{S}(T \cup X_W)$$

$\rightsquigarrow$ suffices to show: $T \cup X_W \sim X_W$



Theorem (F.)

Let M be a simply connected Spin-manifold of dimension at least 6. $(\mathcal{R}^+(M), \mu_W, e_W)$ is a homotopy-commutative, homotopy-associative H-space.

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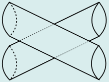

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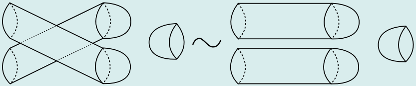

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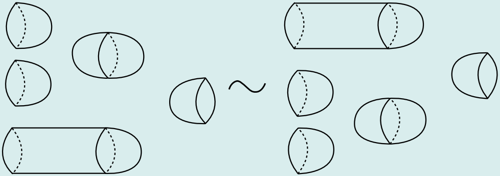

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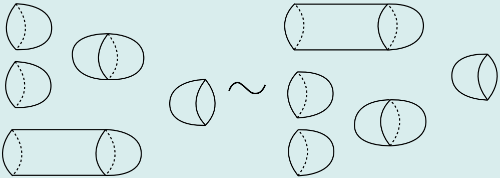

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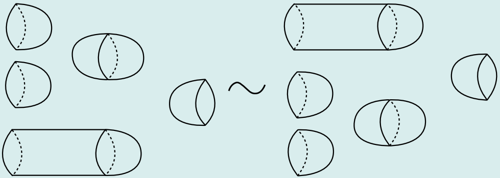

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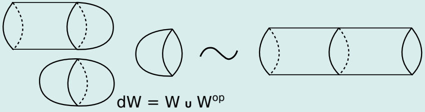

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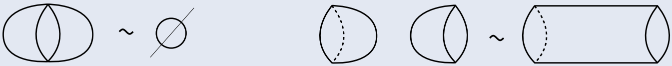

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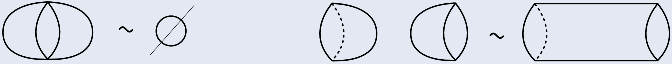

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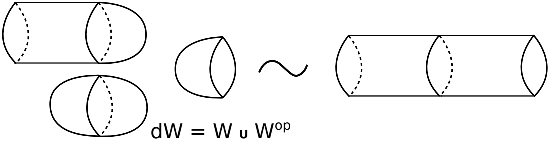

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2. *If $V = W \amalg B$ for B closed with $\alpha(B) \neq 0$, then $\varphi(g) \not\sim g$ for all $g \in \mathcal{R}^+(M)$. In particular $\varphi \neq \mathrm{id}$.*

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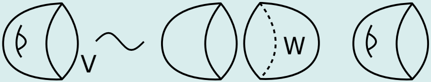

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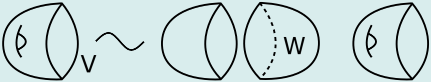

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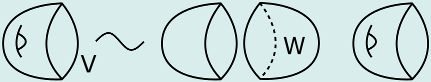

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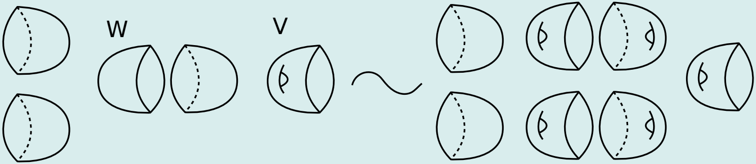

□

Definition

(X, μ, e) H-space, Y a space. An action of X on Y is a homotopy class of a map

$$\rho\colon X \times Y \longrightarrow Y$$

such that $\rho(\mu, \mathrm{id}_Y) = \rho(\mathrm{id}_X, \rho)$ and $\rho(e, \_) = \mathrm{id}$.

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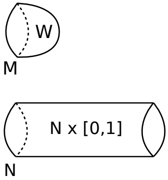

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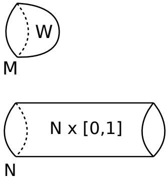

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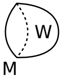

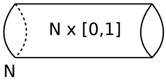

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Proof.

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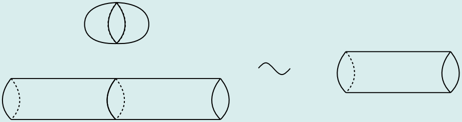

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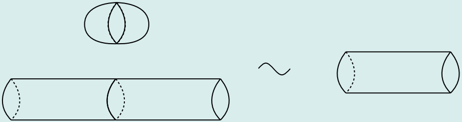

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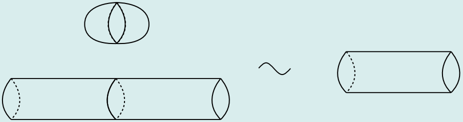

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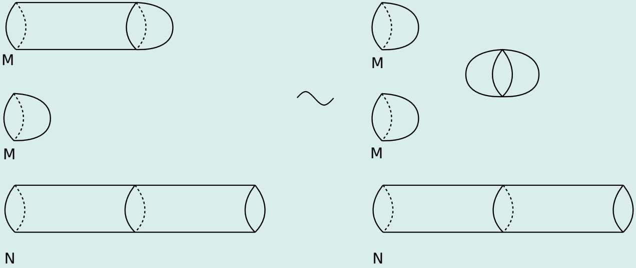

□

Action $\mathrm{Diff}(M) \curvearrowright \mathcal{R}^+(M)$

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If M_0, M_1 are Spin, $\dim M_0, M_1 \geq 6$ and M_1 is simply connected, then there is a map

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In particular: $f^* = \mathcal{S}(M_1 \times [0,1] \amalg T_f)$.

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In particular: If N is Spin-nullcobordant (via $V\colon \emptyset \rightsquigarrow N$) then

$$f^* = \mathrm{id} \quad \Longleftrightarrow \quad f^* e_V \sim e_V$$

Proof.

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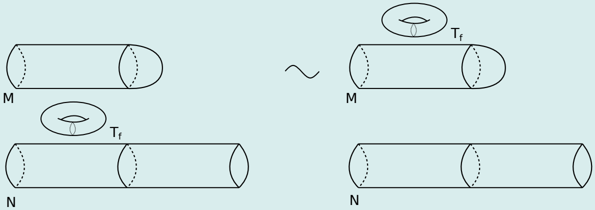

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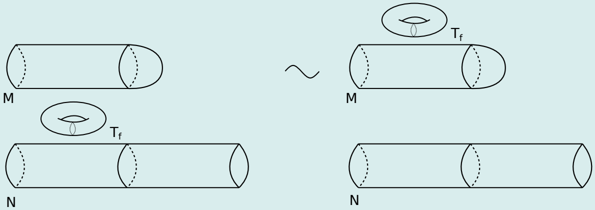

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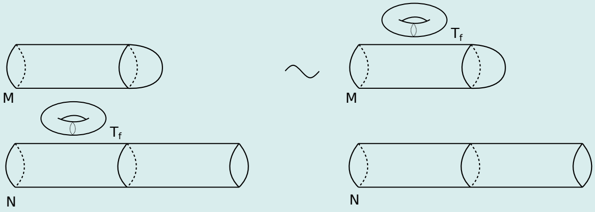

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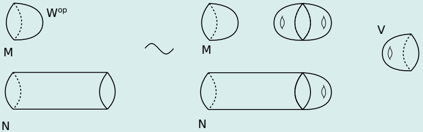
2. 

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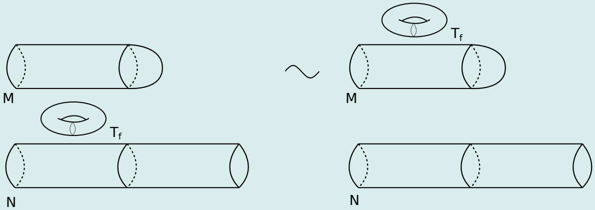

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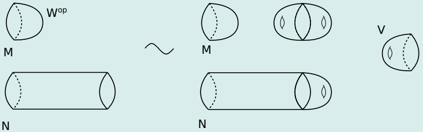
2. 

$\Rightarrow \quad \rho_W = \mu_V \circ (\varphi, \_)$

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$\square$

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Let M be a simply connected Spin-manifold of dimension at least 6. Then $\mathcal{R}^+(M)$ is a homotopy-commutative, homotopy-associative H-space.

Theorem

Let M, N be both simply connected Spin and let $W\colon \emptyset \rightsquigarrow M$, $V\colon \emptyset \rightsquigarrow N$ be Spin-nullbordisms. Then there is an equivalence of H-spaces $\mathcal{R}^+(M) \xrightarrow{\simeq} \mathcal{R}^+(N)$.

Theorem

Let M, N be both simply connected Spin and $W\colon \emptyset \rightsquigarrow M$ (N not necessarily Spin-nullcobordant). Then $\mathcal{R}^+(M) \curvearrowright \mathcal{R}^+(N)$.

Theorem

M, N as before, $f\colon N \to N$ orientation preserving diffeomorphism. If $S(M \times [0,1] \amalg T_f)(e_W) \sim e_W$ then f^* is homotopic to $\mathrm{id}_{\mathcal{R}^+(N)}$.

Theorem

Let M be a simply connected Spin*-manifold [...]*

Theorem

Let M be a simply connected Spin-manifold [...]

Well.. what if it's not?

M manifold.

M manifold.

$\rightsquigarrow$ classifying map $\tau\colon M \to B\mathrm{O}(d)$ of TM

M manifold.

- $\rightsquigarrow$ classifying map $\tau\colon M \to BO(d)$ of TM
- $\rightsquigarrow$ $\exists$ factorization $M \xrightarrow{\ell} B \xrightarrow{\theta} BO(d)$

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  - $\theta\colon B \to BO(d)$ is a 2-coconnected fibration

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$\rightsquigarrow$ $\exists$ factorization $M \xrightarrow{\ell} B \xrightarrow{\theta} B\mathrm{O}(d)$

- $\theta\colon B \to B\mathrm{O}(d)$ is a 2-coconnected fibration
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B is called the <u>Tangential 2-type of M</u>.

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B is called the <u>Tangential 2-type of *M*</u>.

(Example: M Spin-manifold $\rightsquigarrow B = B\mathrm{Spin}(d) \times B\pi_1(M)$.)

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B is called the Tangential 2-type of *M*.

(Example: M Spin-manifold $\rightsquigarrow B = B\mathrm{Spin}(d) \times B\pi_1(M)$.)

$\rightsquigarrow$ Replace Spin by *B*.

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$\rightsquigarrow$ classifying map $\tau \colon M \to B\mathrm{O}(d)$ of TM

$\rightsquigarrow$ $\exists$ factorization $M \xrightarrow{\ell} B \xrightarrow{\theta} B\mathrm{O}(d)$

- $\theta \colon B \to B\mathrm{O}(d)$ is a 2-coconnected fibration
- $\ell \colon M \to B$ is 2-connected.

B is called the Tangential 2-type of M.

(Example: M Spin-manifold $\rightsquigarrow B = B\mathrm{Spin}(d) \times B\pi_1(M)$.)

$\rightsquigarrow$ Replace Spin by B.

Theorem

Let M be a manifold of dimension at least 6 that is nullbordant in its own tangential two type via $W \colon \emptyset \rightsquigarrow M$. Then $(\mathcal{R}^+(M), \mu_W, e_W)$ is a ho-commutative, ho-associative H-space.

M manifold.

$\rightsquigarrow$ classifying map $\tau\colon M \to B\mathrm{O}(d)$ of TM

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(Example: M Spin-manifold $\rightsquigarrow B = B\mathrm{Spin}(d) \times B\pi_1(M)$.)

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Theorem

Let M be a manifold of dimension at least 6 that is nullbordant in its own tangential two type via $W\colon \emptyset \rightsquigarrow M$. Then $(\mathcal{R}^+(M), \mu_W, e_W)$ is a ho-commutative, ho-associative H-space.

$\rightsquigarrow$ applies to $M = F \times S^n$ for every $n \geq 2$ and every F.

Theorem

Let M be a manifold of dimension at least 6 that is nullbordant in its own tangential two type. Then $\mathcal{R}^+(M)$ is a homotopy-commutative, homotopy-associative H-space.

Theorem

Let M, N be manifolds with the same tangential 2-type B and let $W : \emptyset \rightsquigarrow M$, $V : \emptyset \rightsquigarrow N$ be B-nullbordisms. Then there is an equivalence of H-spaces $\mathcal{R}^+(M) \xrightarrow{\simeq} \mathcal{R}^+(N)$.

Theorem

Let M, N manifolds with the same tangential 2-type B and let $W : \emptyset \rightsquigarrow M$ be a B-nullbordism. Then $\mathcal{R}^+(M) \curvearrowright \mathcal{R}^+(N)$.

Thank you for your attention!