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On the Gromov-Lawson surgery theorem

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Georg-Joachim Frenck

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1 Introduction

The main goal of this thesis is to show that the spaces of metrics of positive scalar curvature $\mathcal{R}^+(M)$ and $\mathcal{R}^+(N)$ are homotopy equivalent, if N is obtained from M by surgery in codimension of at least 3 and dimension of at least 2. This result is originally due to Chernysh [3] and has been first published by Walsh [15].

Using this, one can also derive the spin cobordism invariance of $\mathcal{R}^+(M)$. This can be done by making the cobordism 2-connected (this requires a spin cobordism) and then using handle cancellation and the h-cobordism theorem to deduce that one of the boundaries of the cobordism can be obtained from the other one by surgeries that meet our dimension requirements (see [6] & [13]).

In chapter 2 we prepare the proof of the main theorems. We first establish a notation for this thesis and define scalar curvature. Afterwards we define a topology on the space of maps between manifolds and show some basic properties of this topology. After defining the space of metrics $\mathcal{R}(M)$, the space of metrics of positive scalar curvature (psc metrics) $\mathcal{R}^+(M)$ and the space of standard metrics $\mathcal{R}_0^+(M)$, the last part of this chapter will be the proof that isotopic metrics are concordant.

The 3rd chapter will consist of the proof of the originial Gromov-Lawson surgery theorem. We will first explain the geometric idea of the proof and then describe the bending argument. Here we will do a detailed computation of the scalar curvature formula which will show that there is a slight mistake in the curvature formula from [6], [13] and [14].

In chapter 4 we will describe the Gromov-Lawson-Chernysh-deformation, which takes a given metric and deforms it into a standard metric. This will be the main tool to prove the homotopy equivalence described above. It has, however, one flaw: It is not constant on standard metrics, i.e. it cannot be guaranteed that a standard metric remains standard during the deformation.

Therefore we will take a look at warped product metrics in the disc. Here, the main goal is to deform a warped metric into a local torpedo metric, i.e. we want to deform it so that it has a certain standard form inside a small disc.

Chapter 6 will be all about putting the pieces together. We will show that the space of standard metrics $\mathcal{R}_0^+(M)$ is a weak deformation retract of the space of locally warped metrics $W(N,\tau)$ and then use the Gromov-Lawson-Chernysh-deformation to show homotopy equivalence of $W(N,\tau)$ and $\mathcal{R}^+(M)$. Surgery invariance of the homotopy type of $\mathcal{R}^+(M)$ will then come as an easy corollary.

2 Preliminaries

In the first part of this chapter (2.1-2.5) we introduce notation in order to compute the curvature of manifolds. For a more detailed introduction to differential geometry and curvature, see [9]. In (2.6 & 2.9) we prove several lemmata that will prepare the proof of the GROMOV-LAWSON surgery theorem. In (2.7) we define a topology on spaces of mappings between manifolds. This is necessary, as we want to prove topological properties of the space of Riemannian metrics. Subsequently in (2.8) we will define this space and also the space of standard metrics.

2.1 Tensors

2.1.1 Tensors on vector spaces

Details for this section can be found in [9, chapter 2, pp. 11-14].
Let V be an n dimensional vector space.

Definition 2.1. A multilinear map

$$F\colon \underbrace{V^* \times \cdots \times V^*}_{\text{l copies}} \times \underbrace{V \times \cdots \times V}_{\text{k copies}} \to \mathbb{R}$$

is called a (k,l)-tensor or k-covariant, l-contravariant tensor on V. The space of (k,l)-tensors on V will be denoted by $T_l^k(V)$.

Lemma 2.2 ([9, p. 12])**.** *$T_{l+1}^k(V)$ is isomorphic to $\big\{F\colon \underbrace{V^* \times .. \times V^*}_{l} \times \underbrace{V \times .. \times V}_{k} \to V$ multilinear$\big\}$. As a special case, we deduce $T_1^1(V) \cong End(V)$.*

Definition 2.3. The trace of a $(1,1)$-tensors A is defined as

$$\operatorname{tr}(A) = \sum_{i=1}^{n} A(e^i, e_i),$$

where (e_i) is a basis of V and (e^i) is the dual basis of V^*.
Using this, one can define the trace as a map

$$\begin{array}{rcl} \operatorname{tr}\colon T_{l+1}^{k+1}(V) & \to & T_l^k(V) \\ A & \mapsto & \Big((\omega^1, \ldots, \omega^l, V_1, \ldots, V_k) \mapsto \operatorname{tr}\big(\underbrace{A(\omega^1, \ldots, \omega^l, \cdot, V_1, \ldots, V_k, \cdot)}_{\binom{1}{1}\text{-tensor}}\big)\Big). \end{array}$$

Lemma 2.4. *There is a natural product, called the tensor product:*

$$\begin{aligned} \otimes\colon T^k_l(V)\times T^p_q(V) &\to T^{k+p}_{l+q}(V)\\ (F,G) &\mapsto F\otimes G\\ F\otimes G\big(\omega^1,\dots,\omega^{l+q},X_1,\dots,X_{k+p}\big) &:= F(\omega^1,\dots,\omega^l,X_1,\dots,X_k)G(\omega^{l+1},\dots\\ &\quad \dots,\omega^{l+q},X_{k+1},\dots,X_{k+p}). \end{aligned}$$

2.1.2 Tensors on vector bundles

Details for this section can be found in [9, chapter 2, pp. 16-21].
Let M be a smooth manifold, T_pM its tangent space and TM its tangent bundle.

Definition 2.5.

$$\begin{aligned} T^k_l M &:= \coprod_{p\in M} T^k_l(T_pM)\\ \mathcal{T}^k_l(M) &:= \Gamma(M,T^k_lM)=\big\{f\colon M\to T^k_lM \text{ smooth}\big\}. \end{aligned}$$

$\mathcal{T}^k_l(M)$ is called the space of (k,l)-tensor fields. In particular, one sees that $\mathcal{T}^1(M)$ is the space of forms and $\mathcal{T}_1(M) =: \mathcal{T}(M)$ is canonically isomorphic to the space of vector fields.

The trace and the tensor product for tensor fields are defined in analogy to (2.3) and (2.4):

$$\begin{aligned} \mathrm{tr} &: \mathcal{T}^{k+1}_{l+1}(M)\to\mathcal{T}^k_l(M)\\ \otimes &: \mathcal{T}^k_l(V)\times\mathcal{T}^p_q(V)\to\mathcal{T}^{k+p}_{l+q}(V). \end{aligned}$$

Lemma 2.6 ([9, p. 21])**.** *Any (k,l)-tensor field F is multilinear over $C^\infty(M)$ (which will be called C^∞-multilinear), i.e.*

$$F(\dots,f\alpha+g\beta,\dots)=fF(\dots,\alpha,\dots)+gF(\dots,\beta,\dots)$$

for all $f,g\in C^\infty(M)$.
F induces a C^∞-multilinear map

$$\tilde F\colon \underbrace{\mathcal{T}^1(M)\times\dots\times\mathcal{T}^1(M)}_{l}\times\underbrace{\mathcal{T}(M)\times\dots\times\mathcal{T}(M)}_{k}\to C^\infty(M).$$

Lemma 2.7 (**Tensor Characterization Lemma,** [9, p. 21])**.** *A map*

$$\tilde F\colon \mathcal{T}^1(M)\times\dots\times\mathcal{T}^1(M)\times\mathcal{T}(M)\times\dots\times\mathcal{T}(M)\to C^\infty(M)$$

is induced by a (k,l)-tensor field F, if and only if it is C^∞-multilinear. Similarly, a map

$$\tilde F\colon \underbrace{\mathcal{T}^1(M)\times\dots\times\mathcal{T}^1(M)}_{l}\times\underbrace{\mathcal{T}(M)\times\dots\times\mathcal{T}(M)}_{k}\to \mathcal{T}(M)$$

is induced by a $(k,l+1)$-tensor field F, if and only if it is C^∞-multilinear.

2.2 Riemannian metrics

Details for this section can be found in [9, chapter 3, pp. 23-29].
Let M be a smooth manifold.

Definition 2.8. A Riemannian metric on M is a $(2,0)$-tensor field $g \in \mathcal{T}^2 M$, such that:

1. g is symmetric, i.e. $g(X,Y) = g(Y,X)$
2. g is positive definite, i.e. $g(X,X) \geq 0$ and $g(X,X) = 0 \iff X = 0$.

A manifold M together with a Riemannian metric g is called a Riemannian manifold.
If (M,g) and (N,h) are Riemannian manifolds, $M \times N$ has a natural metric $k = g+h$, called the product metric, defined by

$$k(X_1 + X_2, Y_1 + Y_2) = g(X_1, Y_1) + h(X_2, Y_2).$$

For a map $\alpha\colon N \to M$, the pullback $\alpha^* g$ is defined by

$$\alpha^* g(X,Y) := g\big(D\alpha(X), D\alpha(Y)\big),$$

where $D\alpha$ is the differential of α.
A diffeomorphism $\phi\colon (M,g) \to (N,h)$ is called an isometry, if $\phi^* h = g$.

From now on (M,g) is an n-dimensional Riemannian manifold.

Definition 2.9. The musical isomorphism $^\flat$ is defined by

$$\begin{aligned} {}^\flat\colon TM &\to T^*M \\ X &\mapsto \big(Y \mapsto g(X,Y)\big). \end{aligned}$$

In local coordinates $(\partial_1, \dots, \partial_n)$, dual coordinates $(dx^1, \dots, dx^n)$ and $g = (g_{ij})$ we get:

$$X^\flat = g(\sum_i X^i \partial_i, \cdot) = \sum_{i,j} g_{ij} X^i dx^j.$$

Since $^\flat$ is an isomorphism, we get its inverse map

$${}^\sharp\colon T^*M \to TM$$

given by

$$\omega^\sharp = \sum_{i,j} g^{ij} \omega_j \partial_i,$$

where g^{ij} are the coefficients of the inverse matrix g^{-1}.
These two operators can also be applied to tensors of any rank:

$$\begin{aligned} {}^\flat &: \quad \mathcal{T}_l^{k+1}(M) \to \mathcal{T}_{l+1}^{k}(M) \\ {}^\sharp &: \quad \mathcal{T}_{l+1}^{k}(M) \to \mathcal{T}_l^{k+1}(M). \end{aligned}$$

Definition 2.10. The trace of a $(k, 0)$ tensor h for $k \geq 2$ is defined as

$$\mathrm{tr}_g h := \mathrm{tr} h^\sharp.$$

Then $\mathrm{tr}_g h$ is a $(k-2, 0)$ tensor given by:

$$\mathrm{tr}_g h(X_1, \dots, X_{k-2}) = \sum_i h(\partial_i, X_1, \dots, X_{k-2}, \partial_i).$$

2.3 Connections

Details for this section can be found in [9, chapters 4 & 5, pp. 49-76]. In order to talk about curvature on M, one has to define straight lines in M. If $M = \mathbb{R}^n$, straight lines should be given by curves with acceleration identically zero, i.e. curves with constant first derivative. Connections give us a possibility of differentiating vector fields on a manifold. Since the velocity of a curve is a vector field, we can use them to define constant speed curves.

2.3.1 Connections on vector bundles

Definition 2.11. Let $\pi \colon E \to M$ be a vector bundle over a manifold M and let $\mathcal{E}(M)$ denote the space of smooth sections of E.
A connection in E is a map

$$\begin{aligned} \nabla \colon \mathcal{T}(M) \times \mathcal{E}(M) &\to \mathcal{E}(M) \\ (X, Y) &\mapsto \nabla_X Y \end{aligned}$$

satisfying the following properties:

1. $\nabla_X Y$ is C^∞-linear in X, i.e.

$$\nabla_{fX_1 + gX_2} Y = f\nabla_{X_1} Y + g\nabla_{X_2} Y \text{ for } f, g \in C^\infty(M)$$

2. $\nabla_X Y$ is linear in Y, i.e.

$$\nabla_X (aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2 \text{ for } a, b \in \mathbb{R}$$

3. ∇ satisfies the Leibniz-rule (product rule):

$$\nabla_X (fY) = f\nabla_X Y + X(f)Y \text{ for } f \in C^\infty(M).$$

$\nabla_X Y$ will also be called the covariant derivative of Y in the direction of X.

Definition 2.12. A linear connection on a manifold M is a connection on TM.

Remark 2.13 ([9, p. 51])**.** *A linear connection is not a tensor, as it is not C^∞-multilinear.*

Lemma 2.14 ([9, p. 51]). *Let ∇ be a linear connection, and let $X, Y \in \mathcal{T}(U)$, such that $TM|_U$ is trivializable on $U \subset M$ and let $(E_1, \ldots, E_n)$ be a local frame on U, i.e. $(E_1, \ldots, E_n)$ are everywhere linear independent vector fields. Let $X = \sum_i X^i E_i$ and $Y = \sum_i Y^i E_i$. Furthermore let Γ_{ij}^k be the Christoffel symbols, i.e. the smooth functions satisfying:*

$$\nabla_{E_i} E_j = \sum_k \Gamma_{ij}^k E_k.$$

∇ can be expressed in local coordinates as follows:

$$\nabla_X Y = \sum_k \Big(X(Y^k) + \sum_{i,j} X^i Y^j \Gamma_{ij}^k \Big) E_k.$$

2.3.2 Connections on tensor bundles

Lemma 2.15 ([9, p.53]). *Let ∇ be a linear connection on M.*
There is a unique connection in each tensor bundle $T_l^k M$, also denoted by ∇, such that the following conditions are satisfied:

1. *On TM, ∇ agrees with the given connection*
2. *On $T^0 M = C^\infty(M)$, ∇ is given by ordinary differentiation, i.e.*
$$\nabla_X f = X(f)$$
3. *∇ obeys the product rule with respect to tensor products, i.e.*
$$\nabla_X (F \otimes G) = (\nabla_X F) \otimes G + F \otimes (\nabla_X G)$$
4. *∇ commutes with the trace, i.e.*
$$\nabla_X (\mathrm{tr} Y) = \mathrm{tr} \nabla_X Y.$$

Let $\langle .,. \rangle$ denote the pairing of a form and a vector field. ∇ then satisfies the following two properties:

$$\begin{aligned} \nabla_X \langle \omega, Y \rangle &= \langle \nabla_X \omega, Y \rangle + \langle \omega, \nabla_X Y \rangle \\ (\nabla_X F)(\omega^1, \ldots, \omega^l, Y_1, \ldots, Y_k) &= X(F(\omega^1, \ldots, \omega^l, Y_1, \ldots, Y_k)) + \\ &\quad - \sum_j F(\omega^1, \ldots, \nabla_X \omega^j, \ldots, \omega^l, Y_1, \ldots, Y_k) \\ &\quad - \sum_i F(\omega^1, \ldots, \omega^l, Y_1, \ldots, \nabla_X Y_i, \ldots, Y_k). \end{aligned}$$

Lemma 2.16 ([9, p.54]). *The map $\nabla F \colon \underbrace{\mathcal{T}^1(M) \times \cdots \times \mathcal{T}^1(M)}_{l} \times \underbrace{\mathcal{T}(M) \times \cdots \times \mathcal{T}(M)}_{k+1} \to C^\infty(M)$ defined by*

$$\nabla F(\omega^1, \ldots, \omega^l, Y_1, \ldots, Y_k, X) = \nabla_X F(\omega^1, \ldots, \omega^l, Y_1, \ldots, Y_k)$$

is a $(k+1, l)$-tensor field.

Lemma 2.17 ([9, p.57]). *Let ∇ be a linear connection on M. For curve $\gamma\colon I \to M$ let $\mathcal{T}(\gamma)$ denote the vector fields along γ, i.e. the space of smooth maps $V : I \to TM$, such that $V(t) \in T_{\gamma(t)}M$*
∇ determines a unique operator

$$D_t\colon \mathcal{T}(\gamma) \to \mathcal{T}(\gamma)$$

satisfying the following properties:

1. *D_t is linear over $\mathbb{R}$*
2. *$D_t(fV) = \dot{f}V + fD_tV$ for $f \in C^\infty(I)$*
3. *If $\tilde{V}$ is any extension of V to a neighbourhood of $\gamma(I)$,*

$$D_tV = \nabla_{\dot{\gamma}(t)}\tilde{V}.$$

$D_t(V)$ is called the covariant derivative of V along γ.

2.3.3 Geodesics

Definition 2.18. A curve γ is called a geodesic, if $D_t\dot{\gamma} = 0$.

Remark 2.19. *$D_t\dot{\gamma}$ is the acceleration of γ along γ. Therefore, a geodesic is a curve with vanishing acceleration.*

Theorem 2.20 ([9, p.58]). *Let M be a manifold with a linear connection. For any $p \in M$, $V \in T_pM$ and $t_0 \in \mathbb{R}$, there exists an open interval $I \subset \mathbb{R}$, containing t_0 and a geodesic $\gamma_V : I \to M$, such that $\gamma_V(t_0) = p$ and $\dot{\gamma}_V(t_0) = V$. Any two such geodesics agree on their common domain.*

2.3.4 The Levi-Civita connection

Lemma 2.21 ([9, p.67]). *The following conditions are equivalent for a linear connection ∇ on a Riemannian manifold (M, g):*

1. *∇ is compatible with g, i.e. for all $X, Y, Z \in \mathcal{T}(M)$*

$$\nabla_X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

2. *$\nabla g = 0$*
3. *If V, W are vector fields along any curve γ,*

$$\frac{d}{dt}g(V, W) = g(D_tV, W) + g(V, D_tW).$$

Theorem 2.22 (**Fundamental Lemma of Riemannian Geometry**, [9, p.68])**.** *Let (M, g) be a Riemannian manifold.*
Then there exists a unique linear connection ∇ on M, called the Levi-Civita connection being compatible with g and symmetric, i.e. $\nabla_X Y - \nabla_Y X \equiv [X, Y]$, where $[\_,\_]$ denotes the Lie bracket:

$$[X, Y] = X(Y) - Y(X).$$

In local coordinates, the Christoffel symbols of the Levi-Civita connection are given by:

$$\Gamma_{ij}^k = \frac{1}{2} \sum_l g^{kl} \left(\frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{il}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l}\right),$$

where g^{ij} are the components of the inverse matrix g^{-1}.

2.3.5 The exponential map

Definition 2.23.

$$\mathcal{E} := \{V \in TM : \gamma_V \text{ is defined on an interval containing } [0, 1]\}.$$

The exponential map $\exp : \mathcal{E} \to M$ is defined as follows:

$$\exp(V) := \gamma_V(1).$$

Lemma 2.24 ([9, p.76])**.** *For any $p \in M$, there is a neighbourhood $\mathcal{V}$ of the origin in T_pM, such that $\exp : \mathcal{V} \to \exp(\mathcal{V})$ is a diffeomorphism.*

The exponential map allows us to talk about distances on M. If $\varepsilon > 0$ is small enough, we can transport $B(0, \varepsilon) \subset T_pM$ onto M. We write $\|x - y\|_g$ for the distance between x and y with respect to the metric g.

Definition 2.25. Let $N \subset M$ be a closed submanifold. The normal exponential map $exp^\perp : \nu_M^N \to M$ is defined as the restriction of the exponential map to the normal bundle.

Proposition 2.26. *The normal exponential map is a local diffeomorphism.*

Proof. The proof for the previous lemma in [9, p.76] reveals that the differential of the exponential map on a single tangent space at the origin is the identity. Since $\nu_M^N \cong N \times T_xN^\perp$ locally we get canonical identifications $T_{(x,v)}\nu_M^N \cong T_{(x,v)}(N \times T_xN^\perp) \cong T_xN \times T_xN^\perp = T_xM$. So the differential is the same as the one of the usual exponential map, i.e. it is the identity around the origin and thus the normal exponential map is a local diffeomorphism, too. □

2.4 Curvature

Details for this section can be found in [9, chapter 7, pp. 116-124].

2.4.1 The curvature tensor

Definition 2.27. The map $R\colon \mathcal{T}(M) \times \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M)$ defined by

$$R(X,Y,Z) := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

is called the (Riemann) curvature endomorphism.

Proposition 2.28 ([9, p.117])**.** *The curvature endomorphism is a $(3,1)$-tensor field, i.e. it is C^∞-multilinear.*

Definition 2.29. The (Riemann) curvature tensor $Rm \in \mathcal{T}^4(M)$ is defined by

$$Rm\big(X,Y,Z,W\big) := g\big(R(X,Y,Z),W\big).$$

In other words: The curvature tensor is obtained from the curvature endomorphism by applying the musical isomorphisms.

Proposition 2.30 ([9, p.121])**.**

$$\begin{aligned}
Rm(X,Y,Z,W) &= -Rm(Y,X,Z,W) \\
Rm(X,Y,Z,W) &= -Rm(X,Y,W,Z) \\
Rm(X,Y,Z,W) &= Rm(Z,W,X,Y) \\
0 &= Rm(X,Y,Z,W) + Rm(Y,Z,X,W) + Rm(Z,X,Y,W).
\end{aligned}$$

2.4.2 Scalar curvature

Definition 2.31. The Ricci curvature Ric and the scalar curvature κ are defined by

$$\begin{aligned}
Ric(X,Y) &= \operatorname{tr}\big(Rm\big)(X,Y) \\
\kappa &= \operatorname{tr}(Ric) = \operatorname{tr}\big(\operatorname{tr}(Rm)\big).
\end{aligned}$$

If (E_i) is an orthonormal local frame for TM, we get

$$\begin{aligned}
Ric(X,Y) &= \sum_i Rm\big(E_i,X,Y,E_i\big) \overset{(2.30)}{=} \sum_i Rm\big(X,E_i,E_i,Y\big) \\
\kappa &= \sum_{i,j} Rm\big(E_j,E_i,E_i,E_j\big).
\end{aligned}$$

Remark 2.32.

$$\kappa = \sum_{i \neq j} Rm\big(E_j,E_i,E_i,E_j\big) = 2\sum_{i<j} Rm\big(E_j,E_i,E_i,E_j\big)$$

since $Rm\big(E_i,E_i,E_i,E_i\big) = 0$, which can be seen from (2.30).

2.5 Riemannian submanifolds

Details for this section can be found in [9, chapter 8, pp. 131-141].

Definition 2.33. Let $(\tilde{M}, \tilde{g})$ be a Riemannian manifold. A Riemannian submanifold of $(\tilde{M}, \tilde{g})$ is a Riemannian manifold (M, g) together with an embedding $\iota\colon M \hookrightarrow \tilde{M}$, such that $g = \iota^* \tilde{g}$. In this situation $\tilde{M}$ is called the ambient manifold.

Remark 2.34. *Without loss of generality, we may assume that $M \subset \tilde{M}$, so the metric g is just the restriction of $\tilde{g}$ to M. Therefore, in this section g denotes both the metric on M and $\tilde{M}$.*

2.5.1 The second fundamental form

Definition 2.35. Let X, Y be vector fields on M, which are extended arbitrarily to $\tilde{M}$. We have a decomposition

$$\tilde{\nabla}_X Y = (\tilde{\nabla}_X Y)^\top + (\tilde{\nabla}_X Y)^\perp,$$

where $(\tilde{\nabla}_X Y)^\top$ is tangent to M and $(\tilde{\nabla}_X Y)^\perp$ is normal to M. The second fundamental form $\mathrm{I\!I}$ is defined by

$$\begin{aligned} \mathrm{I\!I}\colon \mathcal{T}(M) \times \mathcal{T}(M) &\rightarrow \mathcal{N}(M) \\ (X, Y) &\mapsto \mathrm{I\!I}(X, Y) = (\tilde{\nabla}_X Y)^\perp, \end{aligned}$$

if $\mathcal{N}(M) = \Gamma(M, \nu_{\tilde{M}}^M)$ are the sections of M into the normal bundle $\nu_{\tilde{M}}^M$ of M in $\tilde{M}$.

Lemma 2.36 ([9, p.134])**.** *The second fundamental form is independent of the extensions of X and Y, symmetric in X and Y and C^∞-bilinear.*

Theorem 2.37 (Gauß curvature equation, [9, p.136])**.** *For any $X, Y, Z, W \in \mathcal{T}(M)$ which are extended arbitrarily to $\tilde{M}$ the following equations hold:*

$$\begin{aligned} \tilde{\nabla}_X Y &= \nabla_X Y + \mathrm{I\!I}(X, Y) \\ \tilde{Rm}(X, Y, Z, W) &= Rm(X, Y, Z, W) - g\big(\mathrm{I\!I}(X, W), \mathrm{I\!I}(Y, Z)\big) + g\big(\mathrm{I\!I}(X, Z), \mathrm{I\!I}(Y, W)\big). \end{aligned}$$

Lemma 2.38 ([9, p.138])**.** *For a curve γ in M and a vector field V tangent to M, we have*

$$\tilde{D}_t V = D_t V + \mathrm{I\!I}(\dot{\gamma}, V).$$

Lemma 2.39. *Let $M \subset \tilde{M}$ be totally geodesic, i.e. for every $p \in M, V \in T_pM$, the geodesic γ_V of M is already a geodesic in $\tilde{M}$. Then the second fundamental form of M vanishes everywhere.*

Proof. Let $p \in M, V \in T_pM$ and let γ_V be the geodesic, i.e. $D_t \dot{\gamma}_V = 0$. Since $M \subset \tilde{M}$ is totally geodesic, γ_V is a geodesic in $\tilde{M}$, hence $\tilde{D}_t \dot{\gamma}_V = 0$. Using the Gauß curvature equation for the derivative along a curve (2.38), we see

$$\mathrm{I\!I}(V, V) = \mathrm{I\!I}(V, \dot{\gamma}_V) = \tilde{D}_t V - D_t V = \tilde{D}_t \dot{\gamma}_V - D_t \dot{\gamma}_V = 0.$$

If $V, W \in \mathcal{T}(M)$ one gets:

$$0 = \mathrm{I\!I}(V+W, V+W) = \underbrace{\mathrm{I\!I}(V,V) + \mathrm{I\!I}(W,W)}_{=0} + 2\mathrm{I\!I}(V,W)$$

and thus $\mathrm{I\!I}(V,W) = 0$. $\square$

2.5.2 Riemannian hypersurfaces

Let $\dim M = n$, $\dim \tilde{M} = n+1$ and let N be a unit normal vector field of M in $\tilde{M}$. Since $\mathrm{I\!I}(X,Y) \in \mathcal{N}(M)$, we can write

$$\mathrm{I\!I}(X,Y) = h(X,Y)N,$$

where $h(X,Y) := g(\mathrm{I\!I}(X,Y), N)$.

Definition 2.40. The shape operator s is defined to be the map

$$s \colon \mathcal{T}(M) \quad \to \quad \mathcal{T}(M),$$

such that

$$g\big(X, s(Y)\big) = h(X,Y).$$

Since h is symmetric s is a self adjoint. Let (E_i) be an orthonormal basis of eigenvectors of s, such that $s(E_i) = \lambda_i E_i$. Then the E_i's are called the principal directions and the λ_i's are called the principal curvatures.

Proposition 2.41 ([9, p. 148])**.** *The shape operator s is given by*

$$s(X) = -\tilde{\nabla}_X N.$$

2.6 Riemannian submersions

In this section we gather a few facts from [11] and [2].

Definition 2.42. A map $f \colon M \to B$ is called a Riemannian submersion if f is a submersion and f is an isometry on the horizontal part of the TM .The horizontal part of TM consists of all tangent vectors $X \in T_pM$ which are orthogonal to the tangent space $T_b f^{-1}\big(f(b)\big)$ of the fiber $f^{-1}\big(f(b)\big) =: F_b$. Let $T_{(h)}M$ and $T_{(v)}M$ denote the horizontal and the vertical part of TM.

Definition 2.43. We define the maps

$$\begin{aligned} \mathcal{H} \colon TM &\to T_{(h)}M \\ \mathcal{V} \colon TM &\to T_{(v)}M \end{aligned}$$

as the horizontal and the vertical projection of the tangent spaces of M. Furthermore, if E, F are vector fields, we define

$$\begin{aligned} T_E F &= \mathcal{H}\nabla_{\mathcal{V}E}(\mathcal{V}F) + \mathcal{V}\nabla_{\mathcal{V}E}(\mathcal{H}F) \\ A_E F &= \mathcal{V}\nabla_{\mathcal{H}E}(\mathcal{H}F) + \mathcal{H}\nabla_{\mathcal{H}E}(\mathcal{V}F). \end{aligned}$$

Let (X_i) be an orthonormal local frame for $T_{(h)}M$ and (U_j) be an orthonormal local frame for $T_{(v)}M$. We then define

$$\begin{aligned} N &= \sum_j T_{U_j} U_j \\ \check{\delta} N &= -\sum_i \sum_j \big((D_{X_i} T)_{U_j} U_j, X_i\big) \\ |A|^2 &= \sum_i \sum_j (A_{X_i} U_j, A_{X_i} U_j) \\ |T|^2 &= \sum_i \sum_j (T_{U_j} X_i, T_{U_j} X_i). \end{aligned}$$

Proposition 2.44 ([2, p. 244]). *Let (M, g) be a Riemannian submersion over $(B, \check{g})$ with fiber $(F_b, \hat{g}_b)$ over $b \in B$, where $\hat{g}_b = g|_{F_b}$ and let π denote the projection map. Let $\kappa, \check{\kappa}, \hat{\kappa}$ be the scalar curvatures of the corresponding metrics. Then*

$$\kappa = \check{\kappa} \circ \pi + \hat{\kappa} - |A|^2 - |T|^2 - |N|^2 - 2\check{\delta} N.$$

Lemma 2.45 ([11, p. 461, p.465]). *Let V, W be horizontal and X, Y be vertical vector fields, let $\nabla, \hat{\nabla}$ and $\check{\nabla}$ denote the Levi-Civita connections of M, F and B and let g be the metric of M.*

1. $\nabla_V W = T_V W + \hat{\nabla}_V W$

2. $Rm(X, Y, Y, X) = \check{Rm}(X, Y, Y, X) - 3 \left\| A_Y X \right\|^2$

3. $Rm(V, X, X, V) = g\Big((\nabla_X T)_V V, X\Big) - \left\| T_V X \right\|^2 + \left\| A_X V \right\|^2$

4. $Rm(V, W, W, V) = \hat{Rm}(V, W, W, V) - \left\| T_V W \right\|^2 + g\big(T_W W, T_V V\big)$.

Lemma 2.46. *If F_b is totally geodesic in M, we have $T \equiv 0$. In this case, we have the following inequality for the scalar curvature:*

$$\kappa \geq \check{\kappa} + \hat{\kappa} - 6|A|^2.$$

Proof. $T \equiv 0$ follows immediately from (2.37), (2.39) and (2.45).

For the scalar curvature we compute:

$$\begin{aligned}
\kappa &= \sum_{i,j} Rm(E_i,E_j,E_j,E_i)\\
&= 2\sum_{i<j\le p} Rm(E_i,E_j,E_j,E_i) + 2\sum_{i\le p<j} Rm(E_i,E_j,E_j,E_i)\\
&\quad +2\sum_{p<i<j} Rm(E_i,E_j,E_j,E_i)\\
&\overset{(2.45,\ 2.,\ 3.\ \&\ 4.)}{=} 2\Big(\sum_{i<j\le p} \tilde{Rm}(E_i,E_j,E_j,E_i) - 3\underbrace{\sum_{i<j\le p} \|A_{E_i}E_j\|^2}_{\le 3|A|^2} + \underbrace{\sum_{i\le p<j} \|A_{E_i}E_j\|^2}_{\ge 0}\\
&\quad + \sum_{p<i<j} \hat{Rm}(E_i,E_j,E_j,E_i)\Big)\\
&\ge \check{\kappa} + \hat{\kappa} - 6|A|^2.
\end{aligned}$$

□

2.7 A topology on $C^\infty(M,N)$

Definition 2.47. Let M,N be two smooth manifolds, $(U,\varphi),(V,\psi)$ charts and $K\subset U$ compact, such that $f(K)\in V$, $f\in C^\infty(M,N)$ and $\varepsilon>0$. A subbasis of the C^r-topology is given by the sets $\mathcal{N}^r\big(f,(U,\varphi),(V,\psi),K,\varepsilon\big)$ being defined by:

$$g\in\mathcal{N}\big(f,(U,\varphi),(V,\psi),K,\varepsilon\big) :\Longleftrightarrow \begin{array}{c} g(K)\subset V \text{ and}\\ \big\|D^k(\psi f\varphi^{-1})(x) - D^k(\psi g\varphi^{-1})(x)\big\| < \varepsilon\\ \forall x\in\varphi(K), k\in\{0,\dots,r\}.\end{array}$$

As an abbreviation we define

$$\|f,g\|^k_{\psi,\varphi}(x) := \Big\|D^k(\psi f\varphi^{-1})(x) - D^k(\psi g\varphi^{-1})(x)\Big\|.$$

The C^∞-topology is defined as the union of all C^r-topologies.

Remark 2.48. *The C^∞-topology defines a first-countable space, which can be seen by choosing $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ for ε. So by [4, p. 258] continuity can be verified by the sequence criterion, i.e. α is continuous, if and only if $f_n\to f$ implies $\alpha(f_n)\to\alpha(f)$.*

Proposition 2.49. *If $\|f_n,f\|^k_{\psi,\varphi}(x)\to 0$ for two charts ψ,φ and for all $x\in\varphi(U)$, then it holds for any other charts ψ',φ', provided, $x\in\varphi'(U')$ and $f(\varphi'(x))\in V'$.*

Proof. Since $\|f_n,f\|^k_{\psi,\varphi}(x)\to 0$, there is an n_0, such that $f_n(\varphi'(x))\in V'$ for all $n\ge n_0$

and the following expression is well defined for big n.

$$\begin{aligned}
\|f_n, f\|^k_{\psi',\varphi'}(x) &= \left\|\psi^{-1}\psi f_n\varphi^{-1}\varphi, \psi^{-1}\psi f\varphi^{-1}\varphi\right\|^k_{\psi',\varphi'}(x) \\
&= \left\|D^k\psi'\psi^{-1}\psi f_n\varphi^{-1}\varphi\varphi'^{-1}(x) - D^k\psi'\psi^{-1}\psi f\varphi^{-1}\varphi\varphi'^{-1}(x)\right\| \\
&= \|D^k(\psi'\psi^{-1})D^k(\psi f_n\varphi^{-1})D^k(\varphi\varphi'^{-1})(x) \\
&\quad -D^k(\psi'\psi^{-1})D^k(\psi f\varphi^{-1})D^k(\varphi\varphi'^{-1})(x)\| \\
&= \left\|D^k(\psi'\psi^{-1})\underbrace{\left(D^k(\psi f_n\varphi^{-1}) - D^k(\psi f\varphi^{-1})\right)}_{\longrightarrow 0}D^k(\underbrace{\varphi\varphi'^{-1})(x)}_{\in\varphi(U\cap U')}\right\| \to 0.
\end{aligned}$$

Since the transition maps are smooth, every derivative of them is continuous and thus the entire term converges to 0. □

Proposition 2.50. *If (φ, U) is a chart and $K \subset U$, then we have the following equivalence:*
For any $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$, such that for all $n > n_0$, $\|f_n, f\|^k_{\psi,\varphi}(x) < \varepsilon$ for all $x \in K \iff \|f_n, f\|^k_{\psi,\varphi}(x) \to 0$ for all $x \in K$.

Proof. "$\Rightarrow$" is clear. "$\Leftarrow$" Since $\|f_n, f\|^k_{\psi,\varphi} : \varphi(U) \to \mathbb{R}$ is continuous,

$$(\|f_n, f\|^k_{\psi,\varphi})^{-1}(-\varepsilon, \varepsilon) =: U_n \subset \varphi(U)$$

is open. Since $\|f_n, f\|^k_{\psi,\varphi}(x) \to 0$ for all $x \in \varphi(K)$, there is an $n_x \in \mathbb{N}$, such that, $x \in U_n$ for all $n \geq n_x$. Then $\bigcup\limits_{x\in K} U_{n_x}$ is an open cover of $\varphi(K)$ and since K is compact, there is a number $n_0 \in \mathbb{N}$, such that $\varphi(K) \subset U_{n_0}$ for all $n \geq n_0$. Hence $\|f_n, f\|^k_{\psi,\varphi}(x) < \varepsilon$ for all $n \geq n_0$. □

Proposition 2.51. *We have the following equivalence:*

1. *For all $\varepsilon > 0$, $r \geq 0$, there is an n_0, such that $\|f_n, f\|^k_{\psi,\varphi}(x) < \varepsilon$ for all $n \geq n_0$ and $0 \leq k \leq r$.*
2. *For all $\varepsilon > 0$, $k \geq 0$, there is an n_0, such that $\|f_n, f\|^k_{\psi,\varphi}(x) < \varepsilon$ for all $n \geq n_0$*

Proof. "1. $\Rightarrow$ 2." is obvious.
"2. $\Rightarrow$ 1." Let $r \geq 0$. For every $0 \leq k \leq r$, there is an $n_k \in \mathbb{N}$, such that $\|f_n, f\|^k_{\psi,\varphi}(x) < \varepsilon$ for all $n \geq n_k$. If we then choose $N_0 := \max\limits_{0\leq k\leq r} n_k$ and then

$$\|f_n, f\|^k_{\psi,\varphi}(x) < \varepsilon \text{ for all } n \geq N_0 \text{ and } 0 \leq k \leq r.$$

□

Remark 2.52. *The first two of these three propositions imply that convergence of functions in the C^∞-topology can be verified pointwise and it does not matter which chart we use. By (2.51) we only have to verify that for all $x \in M$ we have $\|f_n, f\|^k_{\psi,\varphi}(x) \to 0$ in some charts ψ, φ and for all k.*

Proposition 2.53. *The map $f \mapsto Df$ is continuous in the C^∞-topology.*

Proof. Let $f_n \in C^\infty(M,N)$ be a sequence converging to f. Since charts for TM and TN are given by the differentials $\big(U \times TM|_U, (\varphi, D\varphi)\big)$, one derives continuity by (2.52) and:

$$\varepsilon \overbrace{>}^{\text{since } f_n \to f} \|f, f_n\|^{k+1}_{\psi,\varphi}(x) = \|Df, Df_n\|^{k}_{D\psi, D\varphi}(x).$$

□

Proposition 2.54. *The map $C^\infty(M,N) \times C^\infty(N,L) \to C^\infty(M,L)$, $f, g \mapsto g \circ f$ is continuous.*

Proof. This follows from the fact that the C^∞-topology we defined is the same as the ∞-jet-topology [8, p. 62] and from [10, p. 68]. Here, this is only proven for the case that f is proper.

The proposition is still true, if f is not proper, but as we only need it for the case of proper maps this proof suffices for us. □

Corollary 2.55. *The pullback map $Emb(M,N) \times \mathcal{R}(N) \to \mathcal{R}(M)$ is continuous.*

Proof. The pullback map factorizes through

$$\begin{array}{rcl} Emb(M,N) \times \mathcal{R}(N) \to & Emb(M,N) \times C^\infty(M, T^2N) & \to \mathcal{R}(M) \\ (f,g) \mapsto & \big(f, g(f)\big) & \mapsto f^*g. \end{array}$$

where $g(f)$ and f^*g are defined as follows: Let $x \in M$, $V_N, W_N \in T_{f(x)}N$ and $V_M, W_M \in T_xM$. Then

$$\begin{aligned} g(f)(x, V_N, W_N) &= (g \circ f(x))\big(V_N, W_N\big) \\ f^*g(x, V_M, W_M) &= g\big(f(x)\big)\big(D_xf \cdot V_M, D_xf \cdot W_M\big) \\ &= \Big(g(f(x)) \circ (D_xf, D_xf)\Big)(V_M, W_M). \end{aligned}$$

These two maps are continuous by (2.53) and (2.54). □

Lemma 2.56. *Let $h\colon C^\infty(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n) \to \mathbb{R}^n$ be a continuous function.*
Then the map

$$\begin{aligned} \sigma\colon C^\infty(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n) &\to C^\infty(\mathbb{R}, \mathbb{R}^n) \\ f &\mapsto \sigma(f) := x, \text{ such that } \dot{x}(t) = f(t, x(t)) \text{ and } x(0) = h(f) \end{aligned}$$

is continuous.

Proof. By (2.52) it is enough to consider the identity as charts. So it suffices to examine $\|\sigma(f_n),\sigma(f)\|^k(t)$. If $x(t)$ is a solution to the above differential equation, we get that $x(t)=\int_0^t f(s,x(s))ds+x(0)$

$$\begin{aligned}
\|\sigma(f_n),\sigma(f)\|^0(t) &= \|\sigma(f_n)(t)-\sigma(f)(t)\| \\
&= \left\|\int_0^t f_n(s,x_n(s))ds + h(f) - \int_0^t f(s,x(s))ds + h(f_n)\right\| \\
&\leq \left\|\int_0^t f_n(s,x_n(s)) - f_n(s,x(s))ds\right\| \\
&\quad \underbrace{+\int_0^t \underbrace{\|f_n(s,x(s))-f(s,x(s))\|}_{\to 0} ds + \underbrace{\|h(f_n)-h(f)\|}_{\to 0}}_{=:a_n\to 0}.
\end{aligned}$$

Since f_n is smooth for every n, it is Lipschitz with constant K_n on the compact subset $[0,t]\subset\mathbb{R}$ and as $f_n\to f$, we get for all $\varepsilon>0$ and n big enough:

$$\begin{aligned}
\|f_n(x)-f_n(y)\| &\leq \underbrace{\|f_n(x)-f(x)\|}_{\to 0} + \|f(x)-f(y)\| + \underbrace{\|f_n(y)-f(y)\|}_{\to 0} \\
&\leq K\|x-y\|+\varepsilon.
\end{aligned}$$

and so $K_n\to K$ and so K_0 can be chosen, such that $\|f_n(x)-f_n(y)\|\leq K_0\|x-y\|$ for all n. Then

$$\begin{aligned}
\varphi_n(s) &:= \|x(t)-x_n(t)\| = \left\|\int_0^t f_n\big(s,x_n(s)\big)-f_n\big(s,x(s)\big)ds\right\| + a_n \\
&\leq \int_0^t K_0\|x(s)-x_n(s)\|ds + a_n = a_n + \int_0^t \varphi(s)ds.
\end{aligned}$$

and Gronwall's lemma then tells us that

$$\|\sigma(f_n),\sigma(f)\|^0(t) = \|x(t)-x_n(t)\| = \varphi_n(t) \leq a_n\exp(K_0 t)\to 0.$$

We can now conclude the lemma by induction.

$$\begin{aligned}
\|\sigma(f_n),\sigma(f)\|^1(t) &= \big\|f_n\big((t,x_n(t)\big)-f\big(t,x(t)\big)\big\| \\
&\leq \underbrace{\big\|f_n\big((t,x_n(t)\big)-f_n\big((t,x(t)\big)\big\|}_{\leq K_0\|x(t)-x_n(t)\|\to 0} + \underbrace{\big\|f_n\big((t,x(t)\big)-f\big((t,x(t)\big)\big\|}_{\to 0}\to 0 \\
\|\sigma(f_n),\sigma(f)\|^k(t) &= \left\|\Big(\frac{d}{dt}\Big)^{k-1}\Big(f_n(t,x_n(t))-f(t,x(t))\Big)\right\| \\
&= \|F\big(f_n,\ldots,f_n^{(k-1)},x_n(t),\ldots,x_n^{(k-1)}(t),t\big) \\
&\quad -F\big(f,\ldots,f^{(k-1)},x(t),\ldots,x^{(k-1)}(t),t\big)\|,
\end{aligned}$$

where F is a polynomial which comes from the derivation. By induction hypothesis we know that $x_n^{(k-1)}$ converges point wise to $x^{(k-1)}$ and so every argument of F converges and since polynomials are continuous we get that the entire right hand side converges to 0. Putting all the pieces together, we obtain

$$\sigma(f_n)\to\sigma(f)$$

and thus the map σ is continuous. $\square$

Remark 2.57. *Lemma (2.56) tells us that the solution of any first order ordinary differential equation depends continuously on the defining equation and the initial value. Since we can translate any higher order ordinary differential equation into a system of several coupled first order differential equations, we get that the solution of any ordinary differential equation depends continuously on the defining equation (or equations) and on the initial values. This also holds if we have ordinary differential equations on manifolds, since we can look at them in local coordinates.*

2.8 The spaces $\mathcal{R}^+(M)$ and $\mathcal{R}_0^+(M)$

Definition 2.58. A torpedo curve of radius ε is a curve in $\mathbb{R}^2$ which starts with a horizontal line segment at $(0,\varepsilon)$ and ends in the arc of a circle of radius ε.

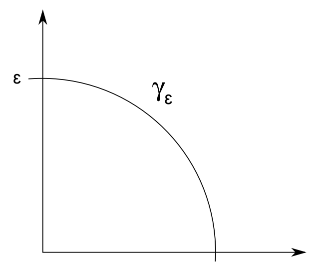


Figure 2.1: A torpedo curve of radius ε

A torpedo metric of radius ε on D^q is the metric induced by the restriction of the euclidian metric on $\mathbb{R}^{q+1}$ to

$$T_{\gamma_\varepsilon} := \{(x,t) \in \mathbb{R}^q \times \mathbb{R} : (\|x\|, t) \in Graph(\gamma_\varepsilon)\} \cong D^q.$$

Remark 2.59. *Sometimes the following is also used as the definition of a torpedo metric:*
A torpedo metric is a metric of positive scalar curvature on D^q, which coincides with the round metric on $S^{q-1} = \partial D^q$ and is the metric of the standard round sphere S^q near the center of D^q.

Definition 2.60. We define

$$\begin{aligned} \mathcal{R}(M) &= \{g \in C^\infty(M, T^2M) : g \text{ is a riemannian metric}\} \\ \mathcal{R}^+(M) &= \{g \in \mathcal{R}(M) : g \text{ has positive scalar curvature}\}. \end{aligned}$$

We equip the space $\mathcal{R}(M)$ with the C^∞-topology and the space $\mathcal{R}^+(M)$ with the subspace topology.

Definition 2.61. Let τ be a tubular neighbourhood of N^p in M^n, let g_0 be a torpedo metric of radius T_0 and let g_N be a metric on N. A metric g is called a standard metric with respect to τ, g_0 and g_N, if

$$\tau^* g = g_N + g_0.$$

We define

$$\begin{aligned} \mathcal{R}_0(M) &= \{g \in \mathcal{R}(M) : g \text{ is a standard metric}\} \\ \mathcal{R}_0^+(M) &= \mathcal{R}_0(M) \cap \mathcal{R}^+(M) \end{aligned}$$

and equip them with the subspace topology. The dependence on τ, g_0 and g_N is omitted in this notation.

Proposition 2.62. *Let $U \subset M$ and let $\partial_{x_1}, \ldots, \partial_{x_n}$ be a local frame for TU. The map*

$$\begin{aligned} \chi_{ijk} \colon \mathcal{R}(M) &\to C^\infty(M, \mathbb{R}) \\ g &\mapsto \Gamma_{ij}^k, \end{aligned}$$

which maps a given metric to its Christoffel symbols (in local coordinates) is continuous.

Proof. In local coordinates, the Christoffel symbols are given by

$$\Gamma_{ij}^k = \frac{1}{2} \sum_l g^{kl} \Big(\frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{il}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l} \Big).$$

By (2.53) we know that the terms of the form $\frac{\partial g_{jl}}{\partial x_i}$ are continuous in g. By Cramer's rule, we have:

$$g^{kl} = \frac{(-1)^{k+l}}{\det(g)} \det(M_{kl}),$$

where M_{kl} is given by omitting the k-th row and the l-th column of g. Since the determinant is given by a polynomial, it is continuous and we can conclude that the given map χ_{ijk} is continuous for every $i, j, k \in \{1, \ldots, n\}$. □

Proposition 2.63. *Let $(\psi, u), (\varphi, V)$ be charts of $\mathcal{R}(M)$ that map fibers to fibers and are fiberwise linear. Then $\mathcal{N}^r\big(f, \varphi, \psi, K, \varepsilon\big)$ is starshaped for all $r \in \mathbb{N}$, with respect to fiberwise addition and multiplication.*

Proof. Let $g \in \mathcal{N}^r\big(f, \varphi, \psi, K, \varepsilon\big)$. First we show that $(tf + (1-t)g)(K) \subset V$, which is clear, because $f(K) \subset U \supset g(K)$ and because addition and multiplication are defined fiberwise.

Now we examine $\|f, tf + (1-t)g\|_{\psi,\varphi}^k$:

$$\begin{aligned} \|f, tf + (1-t)g\|_{\psi,\varphi}^k &= \Big\| D^k\big(\psi f \varphi\big) - D^k\big(\psi(tf + (1-t)g)\varphi\big) \Big\| \\ &= |1-t| \, \Big\| D^k\big(\psi f \varphi\big) - D^k\big(\psi g \varphi\big) \Big\| < \varepsilon. \end{aligned}$$

This completes the proof. □

Proposition 2.64. *Let g_1, g_2 be two metrics on M. Then the maps $I \to \mathcal{R}(M), t \mapsto tg_1$ and $I \to \mathcal{R}(M), t \mapsto (1-t)g_1 + tg_2$ is continuous.*

Proof.

$$\begin{aligned}
&\left\|\big((1-t)g_1 + tg_2\big), \big((1-t)g_1 + t_n g_2\big)\right\|^k_{\psi,\varphi} \\
&= \left\| D^k(\psi \circ \big((1-t)g_1 + tg_2\big) \circ \varphi) - D^k(\psi \circ \big((1-t_n)g_1 + t_n g_2\big) \circ \varphi) \right\| \\
&\leq |(1-t) - (1-t_n)| \left\| D^k(\psi g_1 \varphi) \right\| + |t - t_n| \left\| D^k \psi g_2 \varphi \right\| \to 0.
\end{aligned}$$

The last inequality originates from the fact that the multiplication occurs only in the fibers and we can choose the charts to be fiberwise linear. □

Corollary 2.65. *Let M be compact. If $g_n \to g$ in $\mathcal{R}^+(M)$ for the C^r-topology for some $r \geq 1$, then for some $n \geq 0$ g_n lies in the same path component as g for the C^∞-topology.*

Proof. We have the continuous function

$$\begin{aligned}
\kappa\colon \mathcal{R}(M) &\to \mathbb{R} \\
h &\mapsto \kappa(h) := \min_{x \in M}(\kappa^h(x)),
\end{aligned}$$

where $\kappa^h(x)$ is the scalar curvature of h at the point x. From the definitions we see that the Christoffel symbols depend continuously on the metric and its first derivative, the curvature tensor depends continuously on the Christoffel symbols and the scalar curvature depends continuously on the curvature tensor. We get that this map is continuous in the C^r topology if r is at least one (note that we need the first derivative of the metric to define the Christoffel symbols!).

Since g is a metric of positive scalar curvature and M is compact, we know that $\kappa(g) > a$ for some $a > 0$. We deduce that there is an open neighbourhood U of g in $\mathcal{R}(M)$, such that $\kappa(U) \subset (a, \infty)$ and so every metric in this neighbourhood is a psc metric. Since $g_n \to g$ in the C^r-topology we know that for n big enough $g_n \in U$ and without loss of generality, we may assume that $U = \mathcal{N}^r\big(g, \varphi, \psi, K, \varepsilon\big)$. This is star shaped, which means that the linear path $tg_n + (1-t)g$ lies entirely in $U \subset \mathcal{R}^+(M)$. Thus, we have created a path from g_n to g which is continuous in the C^∞-topology by (2.64). □

2.9 Isotopy implies concordance

Lemma 2.66 ([6, p. 425], [14, p.73]). *We have:*

1. *The principal curvature of an embedded hypersurface $S^{n-1}(\varepsilon)$ are of the form $-\frac{1}{\varepsilon} + O(\varepsilon)$ for small epsilon.*

2. *The term $O(\varepsilon)$ depends on the metric and its derivatives.*

3. *Let g_ε be the induced on $S^{n-1}(\varepsilon)$ and let $g_{0,\varepsilon}$ be the standard euclidian metric of curvature $\frac{1}{\varepsilon^2}$. Then as $\varepsilon \to 0$, $\frac{1}{\varepsilon^2} g_\varepsilon \to \frac{1}{\varepsilon^2} g_{0,\varepsilon} = g_{0,1}$ in the C^2-topology described in (2.7).*

Proof. 1. and 3. are proven in [6, p. 425] and [14, p. 73]. For 2. we are going to examine the computation in these two papers. The following terms are the one, where an $O(.)$ arises:

$$\begin{aligned} g_{ij}(x) &= \delta_{ij} + \sum_{k,l} a_{ij}^{kl} x_k x_l + O(\|x\|^3) \\ \Gamma_{ij}^k &= \sum_l \gamma_{ij}^{kl} + O(\|x\|^2) = O(\|x\|) \\ g(D_t\dot{\gamma}, e_1) &= -\frac{1}{\varepsilon} + \underbrace{\Gamma_{22}^l(\varepsilon, 0, \ldots, 0)}_{=O(\varepsilon)}. \end{aligned}$$

So the $O(\varepsilon)$ only depends on the derivatives of the Christoffelsymbols which depend on the metric and its derivatives. $\square$

Remark 2.67.

$$g(r) \text{ is } O\big(f(r)\big) \iff \exists A, B \in \mathbb{R} : A \cdot f(r) \leq g(r) \leq B \cdot f(r), \text{ for all } r \ll \infty$$

Lemma 2.68 ([6, p.430])**.** *Let $g_t, t \in [0,1]$ be a continuous (in the C^∞-topology) family of metrics on a compact manifold X. If the scalar curvature of g_t is positive for all t, then there exists an $a_0 > 0$, such that for all $a \geq a_0$, the metric*

$$h^a = g_{t/a} + dt^2$$

on $X \times [0,a]$, where dt^2 denotes the standard metric on $\mathbb{R}$, has positive scalar curvature.

Remark 2.69. *This lemma is an elementary way of stating that isotopic metrics are concordant.*

Proof. Let ∇ and $\tilde{\nabla}$ be the Levi Civita connections with respect to g_0 and h^a and let $(x_1, \ldots, x_{n+1})$ be local coordinats at a point $(p,t) \in X \times [0,a]$, where $t = x_{n+1}$. We get:

$$\begin{aligned}
g_{t/a}(x_1,\dots,x_n) &= \sum_{i,j}^{n} \underbrace{(g_{t/a})_{i,j}(x_1,\dots,x_n)}_{=:g_{ij}(x_1,\dots,x_n,t/a)} dx^i dx^j \\
h^a(x_1,\dots,x_{n+1}) &= \sum_{i,j}^{n+1} \gamma^a_{ij}(x_1,\dots,x_{n+1}) dx^i dx^j \\
&= \sum_{i,j}^{n} g_{ij}(x_1,\dots,\frac{x_{n+1}}{a}) dx^i dx^j + dt^2 \\
\gamma^a(x_1,\dots,x_{n+1}) &= \begin{pmatrix} \left(g_{ij}(x_1,\dots,\frac{x_{n+1}}{a})\right)_{i,j} & 0 \\ 0 & 1 \end{pmatrix} \\
(\gamma^a)^{-1}(x_1,\dots,x_{n+1}) &= \begin{pmatrix} \left(g^{ij}(x_1,\dots,\frac{x_{n+1}}{a})\right)_{i,j} & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

We will now compute the Christoffel symbols $(\Gamma^a)^{n+1}_{ij}$ of the Levi-Civita connection with respect to h^a:

$$\begin{aligned}
\frac{\partial \gamma^a_{ij}}{\partial x_l}(x_1,\dots,x_{n+1}) &= \begin{cases} 0, & \text{if } i=n+1 \text{ or } j=n+1 \\ \frac{\partial g_{ij}}{\partial x_l}(x_1,\dots,\frac{x_{n+1}}{a}), & \text{if } i,j,l \le n \\ \frac{1}{a}\frac{\partial g_{ij}}{\partial x_l}(x_1,\dots,\frac{x_{n+1}}{a}), & \text{if } i,j \le n \text{ and } l=n+1 \end{cases} \\
(\Gamma^a)^k_{ij} &= \frac{1}{2}\sum_l (\gamma^a)^{kl}\left(\frac{\partial \gamma^a_{jl}}{\partial x_i} + \frac{\partial \gamma^a_{il}}{\partial x_j} - \frac{\partial \gamma^a_{ij}}{\partial x_l}\right).
\end{aligned}$$

If $i,j \le n$, we have:

$$\begin{aligned}
(\Gamma^a)^{n+1}_{ij} &= \frac{1}{2}\sum_l \underbrace{(\gamma^a)^{n+1,l}}_{=\delta_{n+1,l}}\left(\frac{\partial \gamma^a_{jl}}{\partial x_i} + \frac{\partial \gamma^a_{il}}{\partial x_j} - \frac{\partial \gamma^a_{ij}}{\partial x_l}\right) = \frac{1}{2}\left(\underbrace{\frac{\partial \gamma^a_{j,n+1}}{\partial x_i}}_{=0} + \underbrace{\frac{\partial \gamma^a_{i,n+1}}{\partial x_j}}_{=0} - \frac{\partial \gamma^a_{ij}}{\partial x_{n+1}}\right) \\
&= -\frac{1}{2}\frac{\partial \gamma^a_{ij}}{\partial x_{n+1}} = -\frac{1}{2}\frac{1}{a}\frac{\partial g_{ij}}{\partial x_{n+1}} \\
&= O(\frac{1}{a}) \\
\frac{\partial (\Gamma^a)^{n+1}_{ii}}{\partial x_{n+1}} &= -\frac{1}{2}\frac{1}{a}\frac{\partial}{\partial x_{n+1}}\left(\frac{\partial g_{ii}}{\partial x_{n+1}}\right) = -\frac{1}{2}\frac{1}{a^2}\frac{\partial^2 g_{ii}}{\partial x^2_{n+1}} = O(\frac{1}{a^2}) \\
(\Gamma^a)^{n+1}_{n+1,j} &= -\frac{1}{2}\frac{\partial \gamma^a_{n+1,j}}{\partial x_{n+1}} \equiv 0 \\
(\Gamma^a)^{m}_{n+1,j} &= \frac{1}{2}\sum_l \underbrace{(\gamma^a)^{m,l}}_{=O(1)} \underbrace{\frac{\partial \gamma^a_{jl}}{\partial x_{n+1}}}_{=O(\frac{1}{a})} = O(\frac{1}{a}).
\end{aligned}$$

Since (g_t) is a continuous family, the function $t \mapsto \frac{\partial g_{ij}}{\partial x_{n+1}}(x_1,\dots,x_n,t)$ is continuous on the compact space $[0,1]$, hence it is bounded, which is the reason why $\frac{\partial g_{ij}}{\partial x_{n+1}}$ is $O(1)$.

If we consider $X \subset X \times [0,a]$ as a submanifold, we can use the Gauss curvature equation (2.37) to compute $\tilde{Rm}(\partial_i, \partial_j, \partial_j, \partial_i)$ as long as $i, j \leq n$. In order to do so, we need an estimate for the second fundamental form of X in $X \times [0,a]$. Let $i, j \leq n$.

$$\begin{aligned} \mathrm{I\!I}(\partial_i, \partial_j) &= (\tilde{\nabla}_{\partial_i}\partial_j)^{\perp} = h^a\big(\sum_k (\Gamma^a)^k_{ij}\partial_k, \partial_{n+1}\big) \cdot \partial_{n+1} \\ &= (\Gamma^a)^{n+1}_{ij} \cdot \partial_{n+1} = O(\frac{1}{a}) \cdot \partial_{n+1}. \end{aligned}$$

Applying the Gauss curvature equation (2.37), we get for $i, j \leq n$:

$$\begin{aligned} \tilde{Rm}(\partial_i, \partial_j, \partial_j, \partial_i) &= Rm(\partial_i, \partial_j, \partial_j, \partial_i) + g\big(\mathrm{I\!I}(\partial_i, \partial_j), \mathrm{I\!I}(\partial_i, \partial_j)\big) - g\big(\mathrm{I\!I}(\partial_i, \partial_i), \mathrm{I\!I}(\partial_j, \partial_j)\big) \\ &= Rm(\partial_i, \partial_j, \partial_j, \partial_i) + O(\frac{1}{a^2}) \end{aligned}$$

So what's left is to compute $\tilde{Rm}(\partial_i, \partial_{n+1}, \partial_{n+1}, \partial_i) = \tilde{Rm}(\partial_{n+1}, \partial_i, \partial_i, \partial_{n+1})$.

$$\begin{aligned} \tilde{Rm}(\partial_{n+1}, \partial_i, \partial_i, \partial_{n+1}) &= g\Big(\tilde{R}(\partial_{n+1}, \partial_i, \partial_i), \partial_{n+1}\Big) \\ &= g\Big(\big(\tilde{\nabla}_{\partial_{n+1}}\tilde{\nabla}_{\partial_i}\partial_i - \tilde{\nabla}_{\partial_i}\tilde{\nabla}_{\partial_{n+1}}\partial_i - \tilde{\nabla}_{\underbrace{[\partial_{n+1}, \partial_i]}_{=0}}\partial_i\big), \partial_{n+1}\Big) \\ &= g\Big(\big(\tilde{\nabla}_{\partial_{n+1}}\sum_k (\Gamma^a)^k_{ii}\partial_k - \tilde{\nabla}_{\partial_i}\sum_k (\Gamma^a)^k_{n+1,i}\partial_k\big), \partial_{n+1}\Big). \end{aligned}$$

$$\begin{aligned} \tilde{\nabla}_{\partial_{n+1}}\sum_k (\Gamma^a)^k_{ii}\partial_k &\overset{(2.14)}{=} \sum_k \Big(\frac{\partial (\Gamma^a)^k_{ii}}{\partial x_{n+1}} + \sum_m (\Gamma^a)^m_{ii}(\Gamma^a)^k_{n+1,m}\Big)\partial_k \\ \tilde{\nabla}_{\partial_i}\sum_k (\Gamma^a)^k_{n+1,i}\partial_k &\overset{(2.14)}{=} \sum_k \Big(\frac{\partial (\Gamma^a)^k_{n+1,i}}{\partial x_i} + \sum_m (\Gamma^a)^m_{n+1,i}(\Gamma^a)^k_{i,m}\Big)\partial_k. \end{aligned}$$

Hence,

$$\begin{aligned} \tilde{Rm}(\partial_{n+1}, \partial_i, \partial_i, \partial_{n+1}) &= \underbrace{\frac{\partial (\Gamma^a)^{n+1}_{ii}}{\partial x_{n+1}}}_{=O(\frac{1}{a^2})} - \underbrace{\frac{\partial (\Gamma^a)^{n+1}_{n+1,i}}{\partial x_i}}_{=0} \\ &\quad + \sum_m \Big[(\Gamma^a)^m_{ii}\underbrace{(\Gamma^a)^{n+1}_{n+1,m}}_{=0} - \underbrace{(\Gamma^a)^m_{n+1,ix}(\Gamma^a)^{n+1}_{m,i}}_{=O(\frac{1}{a^2})}\Big] = O(\frac{1}{a^2}). \end{aligned}$$

Overall we see:

$$\begin{aligned} \kappa^{X\times[0,a]} &= \sum_{i,j\leq n+1} \tilde{Rm}(\partial_i, \partial_j, \partial_j, \partial_i) \\ &= \sum_{i,j\leq n} \tilde{Rm}(\partial_i, \partial_j, \partial_j, \partial_i) + 2\sum_{j=1}^{n} \tilde{Rm}(\partial_{n+1}, \partial_j, \partial_j, \partial_{n+1}) \\ &= \sum_{i,j\leq n} Rm(\partial_i, \partial_j, \partial_j, \partial_i) + O(\frac{1}{a^2}) = \kappa^X + O(\frac{1}{a^2}). \end{aligned}$$

Thus, if a is big enough, $\kappa^{X\times[0,a]}$ is positive on an open neighbourhood of $p \times [0,a]$, since our computation did not depend on the value of t. Since X is compact we can cover it by a finite number of those open subsets, each one with perhaps a different constant a. We then choose the biggest of these a and we get that $X \times [0, a_{max}]$ has positive scalar curvature. □

Remark 2.70. *From the equation*

$$\kappa^{X\times[0,a]} = \kappa^X + O(\frac{1}{a^2})$$

we see that the same proof works, if we replace the condition "positive scalar curvature" by "scalar curvature greater than $B \in \mathbb{R}$*".*

3 The Gromov-Lawson surgery theorem

The construction in this chapter originates from [6].

Theorem 3.1 (Gromov-Lawson surgery theorem, [6, p. 423])**.** *Let N be obtained from M by surgery in codimension at least 3. Then:*

$$\mathcal{R}^+(M) \neq \emptyset \Rightarrow \mathcal{R}^+(N) \neq \emptyset.$$

Remark 3.2. *We will use the term "psc manifold" as a synonym for "manifold, which carries a metric of positive scalar curvature".*

The assumption of codimension at least 3 is vital. If it is dropped, one easily finds the counterexample of the 2-Torus, which is obtained from the 2-sphere by surgery in dimension 0, i.e. codimension 2. However, the 2-sphere is a psc manifold, whereas the 2-torus is not. This follows from [7, p. 210].

3.1 Outline of the proof

The strategy of this proof is to use the flexibility of surgery. Since surgery is only defined up to diffeomorphism, one can stretch and bend the manifold. So in a way, we do not search a metric of positive scalar curvature, but rather take a given metric and deform the manifold until it is a psc manifold.

So, let M^n be the manifold from (3.1) and let S^p be the embedded surgery sphere with trivial normal bundle which can transported to M, using the exponential map. So without loss of generality, we may say that we have an embedding $S^p \times D^q \hookrightarrow M$, where $q = n - p \geq 3$. We identify $S^q \times D^q$ with its image under the embedding.

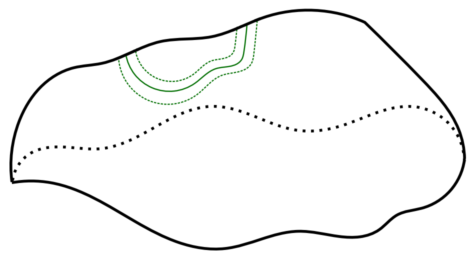


Figure 3.1: A part of the manifold with embedded $S^p \times D^q$

The goal is to replace $S^p \times D^q$ by $D^{p+1} \times S^{q-1}$. We do not take a standard disk D^{p+1} but we put a cylinder in between and perform the surgery on top of it. This will give us the opportunity of changing the metric on the cylinder using (2.68).

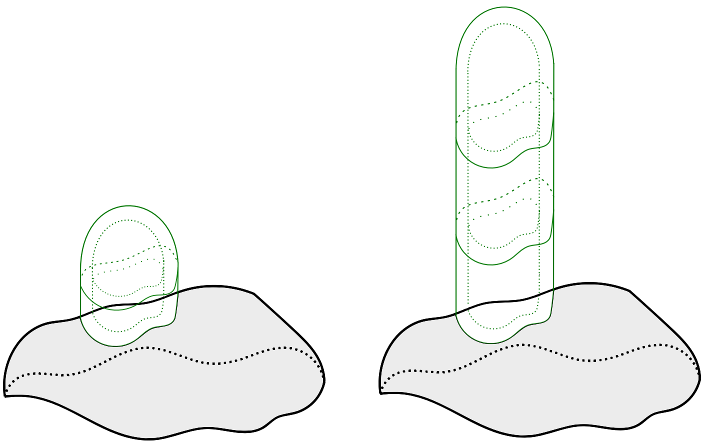


Figure 3.2: Inserting a cylinder

Our goal is to make the scalar curvature on the S^{q-1} big enough, so that we can use (2.46) to estimate the scalar curvature on $S^p \times S^{q-1}$. This means decreasing the radius of D^q. So we won't take a straight cylinder, but we bend the edge near M.

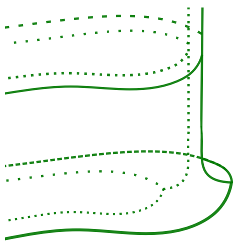


Figure 3.3: Bending the cylinder near the manifold M

So we get that our cylinder starts in M and ends at $S^p \times S^{q-1}(\varepsilon)$, where $\varepsilon > 0$ is the diameter of the normal sphere. The metric on $S^{q-1}(\varepsilon)$ converges to the standard round metric (2.66) and by (2.65), we can homotope the metric on $S^p \times S^{q-1}(\varepsilon)$ through psc metrics to one, where the second factor carries the standard round metric of radius ε, if ε is small enough.

Let g_t be a straight line from the metric on $S^p \times S^{q-1}(\varepsilon)$ to the product metric of two standard round spheres. Then $S^p \times S^{q-1}(\varepsilon)$ is a Riemannian submersion with totally geodesic fibre $S^{q-1}(\varepsilon)$ (Note that the metric on $S^{q-1}(\varepsilon)$ is the same everywhere). This means we can use the estimate (2.46) derived from the O'Neill formulae [11]:

$$\kappa_t^{S^p\times S^{q-1}(\varepsilon)} \geq \underbrace{\kappa_t^{S^{q-1}(\varepsilon)}}_{\kappa^{S^{q-1}(\varepsilon)}\gg 0} + \underbrace{\kappa_t^{S^p} - 6|A_t|^2}_{\geq C} > 0. \tag{3.3}$$

The last part can be interpreted as a smooth function in t over the compact set $[0,1]$ which takes its minimum and maximum, hence it is bounded from below by some $C \in \mathbb{R}$. If ε is small enough, and thus $\kappa^{S^{q-1}(\varepsilon)}$ is big enough, we can maintain $\kappa_t^{S^p\times S^{q-1}(\varepsilon)} > 0$ for all t. Putting these two homotopies together, we get a homotopy from the metric on $S^p \times S^{q-1}(\varepsilon)$ at the end of the cylinder to the product metric of two standard spheres.

Using (2.68) we can put the metric $h^a = g_{t/a} + dt^2$, belonging to the above homotopy on the cylinder, for some big enough a. We then achieved that the cylinder ends on $S^p \times S^{q-1}(\varepsilon)$, where both spheres carry the standard round metric. We then glue in $D^{p+1} \times S^{q-1}(\varepsilon)$, where we regard D^{p+1} as the upper hemisphere of S^{p+1}. Since $S^p \subset S^{p+1}$ is an isometric inclusion, i.e. the metric of $S^{q+1}|_{S^q}$ is the same as the standard round metric of S^q, we get that the metrics on $D^{p+1} \times S^{q-1}$ and the one on the end of the cylinder agree on the boundary.

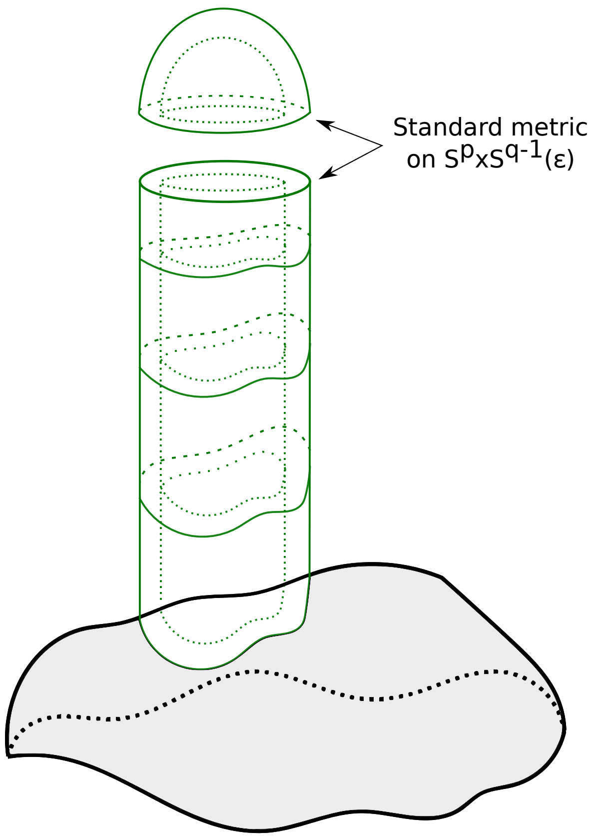


Figure 3.4: Inserting a cylinder

So the metrics can be glued together as well and we get that the new manifold, obtained by this surgery has a psc metric.

If N is obtained from M by surgery, we know that there is a diffeomorphism from N to a manifold N', which is obtained from M by surgery like the one described above. We then take the metric on N to be the pullback of the metric on N' under this diffeomorphism and deduce that there is a psc metric on N.

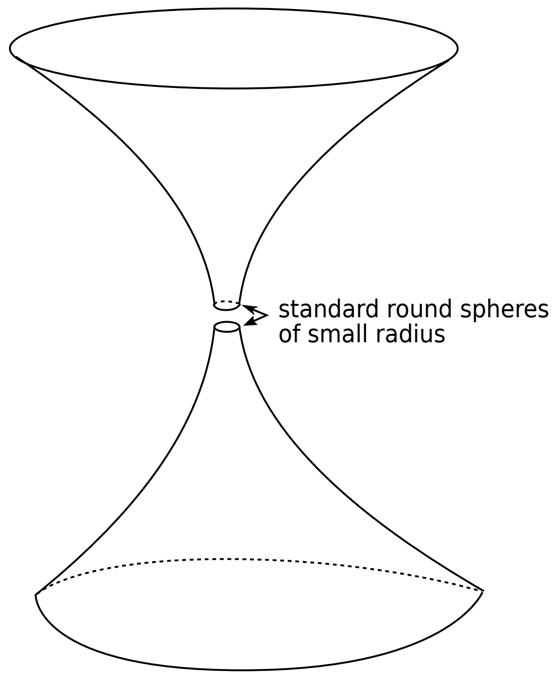
Remark 3.4. *The above estimate (3.3) also works, if we take any manifold N instead of S^p and take fixed metric g_N on N. In this case, we derive:*

Corollary 3.5. *Let g_N be a fixed metric on N. If g is an arbitrary psc metric on M restricted to $N \times S^{q-1}(\varepsilon)$, then for some small enough ε, there is a homotopy through psc metrics from g to $g_N + g_\varepsilon$, where g_ε is the standard round metric of radius ε.*

Remark 3.6. *We did not yet use the vital assumption of $n - p \geq 3$. So there has to be a point in the proof, where it is hard to make the scalar curvature remain positive. Since the bending is the only thing, we haven't done yet, we may conclude that it has to show up there. The bending will be described in the next section.*

Corollary 3.7 ([6, p. 423])**.** *The connected sum of two psc manifolds of dimension a least 3 is a psc manifold.*

Proof. Let M^n be a manifold as above and let $D^n \to M$ be an embedding. After cutting out the interior of D^n and performing the same procedure described above, we arrive at the following situation:



On the boundary, the metric is the standard round metric of a sphere (of small radius). If we do the same thing with the other manifold we get the same thing twice and we can glue these manifolds and their metrics together, since the metrics coincide on the boundary. $\square$

3.2 The bending argument

We will describe the bending by using a bending map $\gamma\colon [0,\infty) \to [0,\infty)\times(0,\overline{r})$, where $\overline{r}$ is the radius of the embedded q-disc. We then define $M_\gamma \subset S^p \times D^q \times \mathbb{R}$ by

$$\begin{aligned}(x,y,t) \in M_\gamma \quad &\Longleftrightarrow \quad (\|y\|_g, t) \in \mathrm{Graph}(\gamma)\\ &\Longleftrightarrow \quad \exists a \in [0,\infty), \text{ such that } \gamma(a) = (\|y\|_g, t)\end{aligned}$$

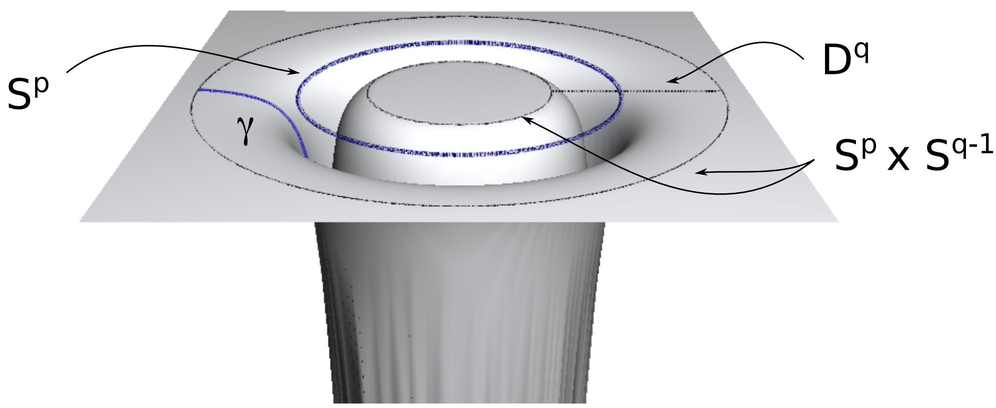


Figure 3.5: Describing the bending

where $\|y\|_g$ denotes the distance from y to the origin of D^q. M_γ is the result of the bending, i.e. the bent cylinder that sits on top of the manifold M.

Now we have the task to choose γ and we have to be very careful in doing so. We want to have γ satisfying the following conditions:

1. γ starts with a vertical line segment (i.e. $t \equiv 0$ in the beginning).
2. γ ends with a horizontal line segment (i.e. $r \equiv r_\infty$, for some $0 < r_\infty \ll \infty$ in the end).
3. M_γ has positive scalar curvature with respect to the induced metric it receives as a submanifold of $S^p \times D^q \times \mathbb{R}$.

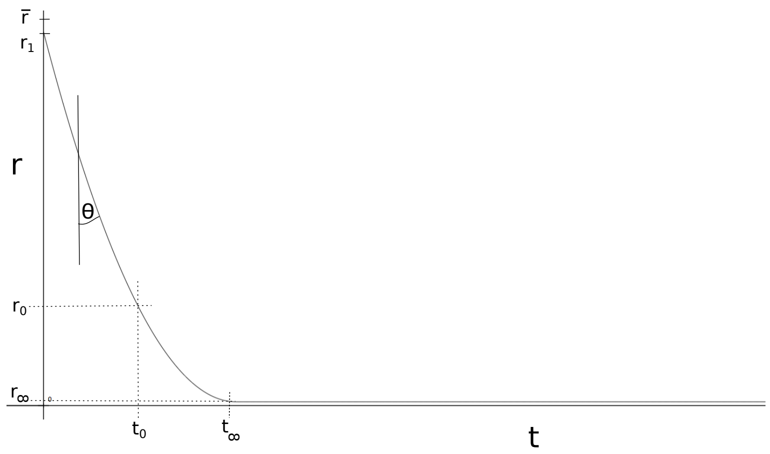


Figure 3.6: The bending map γ

The first condition implies that near $r = \overline{r}$, M_γ is isometric to a portion of M.
The second condition implies that for $t \gg 0$, M_γ is isometric to $S^p \times S^{q-1}(r_\infty) \times \mathbb{R}$, where $S^{q-1}(r_\infty)$ is an embedded q-sphere of small radius.
The third condition is the hard part of choosing γ.

3.2.1 The curvature formula

In order to make the scalar curvature positive, one needs to compute it first. This will be done in the following section.

Lemma 3.8. *Let k be the curvature of γ, i.e. let k be the smooth function satisfying*

$$\ddot{\gamma}(s) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \dot{\gamma}(s) \cdot k(s).$$

The scalar curvature of M_γ is given by:

$$\begin{aligned} \kappa^{M_\gamma} = \kappa^{S^p \times D^q} \quad &+ \quad \sin^2\theta \cdot O(1) + k \cdot \sin\theta \cdot \Big(-\frac{2 \cdot (q-1)}{r} + O(1)\Big) \\ &+ \quad \sin^2\theta \cdot \Big(\frac{(q-1)(q-2)}{r^2} + \frac{2 \cdot (q-1)}{r} \cdot O(1)\Big) + \sin\theta \cdot k \cdot (q-1) \cdot O(r). \end{aligned}$$

Remark 3.9. *This formula is different from the formulae found in the following papers: [6], [13] and [14]. There is a detailed, yet slightly wrong computation in the appendix of [14, pp. 77-80]. We will follow this computation.*

Remark 3.10. *In this formula we see the necessity of $q \geq 3$: For small r the right hand side is dominated by the $\sin^2(\theta)\frac{(q-1)(q-2)}{r^2}$ term, so this one must not vanish, which means that $(q-1)(q-2)$ must be greater than 0. So q must be at least 3 or $q = 0$. The latter case is not very interesting.*

Remark 3.11. *From (2.66) and from the proof of (3.8) we see that the $O(.)$-terms only depend on the metric on M and its derivatives. So there is a constant $C > 0$, such that for all $O(.)$ originating from the formula above the following inequalities hold for r small enough:*

$$\begin{aligned} -C \leq \quad & O(1) \quad \leq C \\ -C \cdot r \leq \quad & O(r) \quad \leq C \cdot r \\ -C \cdot r^2 \leq \quad & O(r^2) \quad \leq C \cdot r^2. \end{aligned}$$

The constant C only depends on the metric we start with and its derivatives and can be chosen to depend continuously (in the C^∞-topology) on the given metric.

Proof of Lemma 3.8. Let $(x, y, t) \in M_\gamma$. First we show that the direction tangent to the curve γ is a principal direction of M_γ in $S^p \times D^q \times \mathbb{R}$. Let l be a geodesic ray in D^q connecting $(x, 0)$ and (x, y). By γ_l we denote the curve $M_\gamma \cap l \times \mathbb{R}$ and we take $\dot{\gamma}_l$ as the tangent vector belonging to this curve. Let s^{M_γ} and s^{γ_l} be the shape operators of M_γ and γ_l in $S^p \times D^q \times \mathbb{R}$ and $l \times \mathbb{R}$. Both of these shape operators shall be chosen with respect to the outward pointing unit vector field η. By [14, p. 77] we know that η is tangential to $l \times \mathbb{R}$, hence it can be used as a normal vector field for γ_l in $l \times \mathbb{R}$.

$$\begin{aligned} s^{M_\gamma}(\dot{\gamma}_l) &= -\nabla^{S^p \times D^q \times \mathbb{R}}_{\dot{\gamma}_l} \eta \\ &= (-\nabla^{S^p \times D^q \times \mathbb{R}}_{\dot{\gamma}_l} \eta)^\top + \underbrace{(-\nabla^{S^p \times D^q \times \mathbb{R}}_{\dot{\gamma}_l} \eta)^\perp}_{=0} \\ &= -\nabla^{l \times \mathbb{R}}_{\dot{\gamma}_l} \eta = s^{\gamma_l}(\dot{\gamma}_l). \end{aligned}$$

Here, $(-\nabla^{S^p \times D^q \times \mathbb{R}}_{\dot{\gamma}_l} \eta)^\top$ shall be the part of $-\nabla^{S^p \times D^q \times \mathbb{R}}_{\dot{\gamma}_l} \eta$ which is tangent to $l \times \mathbb{R}$. This, combined with the fact that $l \times \mathbb{R}$ is totally geodesic in $S^p \times D^q \times \mathbb{R}$, explains, why $(-\nabla^{S^p \times D^q \times \mathbb{R}}_{\dot{\gamma}_l} \eta)^\perp = 0$ (see 2.39). Since $T_{(x,y,t)}\gamma_l$ is a one dimensional subspace, $\dot{\gamma}_l$ is an Eigenvector of s^{γ_l} and whence a principal direction of M_γ. Its principal curvature is the curvature k of γ as l was chosen as a geodesic ray.

We now take an orthonormal basis $(e_1, \ldots, e_n)$ of principal directions of M_γ, such that $e_1 = \dot{\gamma}_l$, $(e_2, \ldots, e_q)$ are tangent to $S^{q-1}(r)$ and $(e_{q+1}, \ldots, e_n)$ are tangent to S^p. Furthermore, we can decompose $\eta = \cos(\theta)\partial_t + \sin(\theta)\partial_r$, where ∂_t denotes the $\mathbb{R}$ direction and ∂_r is the radial direction.

Next, we will compute the principal curvatures $(\lambda_1, \dots, \lambda_n)$ of $(e_1, \dots, e_n)$. The principal curvature of e_1 is k. Since $(e_2, \dots, e_n)$ are eigenvectors of s^{M_γ}, we have:

$$\begin{aligned}
\lambda_j &= \lambda_j \cdot g(e_j, e_j) = g(\lambda_j e_j, e_j) \\
&= g\big(s^{M_\gamma}(e_j), e_j\big) \overset{(2.41)}{=} g\big(-\nabla^{S^p\times D^q\times\mathbb{R}}_{e_j}\eta, e_j\big) \\
&= g\big(-\nabla^{S^p\times D^q\times\mathbb{R}}_{e_j}(\cos\theta\cdot\partial_t), e_j\big) + g\big(-\nabla^{S^p\times D^q\times\mathbb{R}}_{e_j}(\sin\theta\cdot\partial_r), e_j\big).
\end{aligned}$$

Since ∂_t and θ are constant in every e_j direction, we get:

$$\begin{aligned}
\nabla^{S^p\times D^q\times\mathbb{R}}_{e_j}(\cos\theta\cdot\partial_t) &= \partial_j\cos\theta\cdot\partial_t + \cos\theta\cdot\nabla^{S^p\times D^q\times\mathbb{R}}_{e_j}\partial_t = 0 \\
\nabla^{S^p\times D^q\times\mathbb{R}}_{e_j}(\sin\theta\cdot\partial_r) &= \partial_j\sin\theta\cdot\partial_r + \sin\theta\cdot\nabla^{S^p\times D^q\times\mathbb{R}}_{e_j}\partial_r \\
&= \sin\theta\cdot\nabla^{S^p\times D^q\times\mathbb{R}}_{e_j}\partial_r = -\sin\theta\cdot\lambda_j^{S^p\times S^{q-1}},
\end{aligned}$$

where $\lambda_j^{S^p\times S^{q-1}}$ is the principal curvature of e_j as a principal direction of $S^p\times S^{q-1}$ in $S^p\times D^q$. The last equality follows from the fact that ∂_r is normal to $S^p\times S^{q-1}$.

We get for all $j\geq 2$:

$$\lambda_j = \sin\theta\cdot\lambda_j^{S^p\times S^{q-1}}$$

and $\lambda_2^{S^p\times S^{q-1}}, \dots, \lambda_q^{S^p\times S^{q-1}} = -\frac{1}{r} + O(r)$ (2.66) and $\lambda_{q+1}^{S^p\times S^{q-1}}, \dots, \lambda_n^{S^p\times S^{q-1}} = O(1)$, as the curvature of S^p is bounded. All in all, we get:

$$\lambda_j = \begin{cases} k & \text{if } j = 1 \\ \sin\theta\cdot\big(-\frac{1}{r} + O(r)\big) & \text{if } 2\leq j\leq q \\ \sin\theta\cdot O(1) & \text{if } q+1\leq j\leq n. \end{cases}$$

Now we are able to compute the scalar curvature. Since $(e_1, \dots, e_n)$ are principal directions, with the notation of (2.40) $\mathrm{I\!I}(e_i, e_j) = h(e_i, e_j)\eta = g\big(e_i, s^{M_\gamma}(e_j)\big)\eta = \lambda_j g(e_i, e_j)\eta = \delta_{ij}\lambda_j\cdot\eta$. And by the Gauss curvature equation (2.37):

$$\begin{aligned}
Rm^{M_\gamma}(e_i, e_j, e_j, e_i) &= Rm^{S^p\times D^q\times\mathbb{R}}(e_i, e_j, e_j, e_i) + g\big(\mathrm{I\!I}(e_i, e_i), \mathrm{I\!I}(e_j, e_j)\big) - \underbrace{g\big(\mathrm{I\!I}(e_i, e_j), \mathrm{I\!I}(e_i, e_j)\big)}_{=0} \\
&= Rm^{S^p\times D^q\times\mathbb{R}}(e_i, e_j, e_j, e_i) + \lambda_i\lambda_j.
\end{aligned}$$

As $(\partial_r, e_2, ..., e_n)$ form a basis of the tangent space of $S^p\times D^q$, we have for $2\leq i, j\leq n$ that

$$Rm^{S^p\times D^q\times\mathbb{R}}(e_i, e_j, e_j, e_i) = Rm^{S^p\times D^q}(e_i, e_j, e_j, e_i)$$

and

$$\begin{aligned}
Rm^{S^p\times D^q\times\mathbb{R}}(e_1,e_j,e_j,e_1) &= Rm^{S^p\times D^q\times\mathbb{R}}(-\cos\theta\cdot\partial_r+\sin\theta\cdot\partial_t,e_j,e_j,-\cos\theta\cdot\partial_r+\sin\theta\cdot\partial_t)\\
&= Rm^{S^p\times D^q\times\mathbb{R}}(-\cos\theta\cdot\partial_r,e_j,e_j,-\cos\theta\cdot\partial_r)\\
&\quad +Rm^{S^p\times D^q\times\mathbb{R}}(-\cos\theta\cdot\partial_r,e_j,e_j,\sin\theta\cdot\partial_t)\\
&\quad +Rm^{S^p\times D^q\times\mathbb{R}}(\sin\theta\cdot\partial_t,e_j,e_j,-\cos\theta\cdot\partial_r)\\
&\quad +Rm^{S^p\times D^q\times\mathbb{R}}(\sin\theta\cdot\partial_t,e_j,e_j,\sin\theta\cdot\partial_t)\\
&= \cos^2\theta\cdot Rm^{S^p\times D^q\times\mathbb{R}}(\partial_r,e_j,e_j,\partial_r)\\
&= \cos^2\theta\cdot Rm^{S^p\times D^q}(\partial_r,e_j,e_j,\partial_r)\\
&= (1-\sin^2\theta)\cdot Rm^{S^p\times D^q}(\partial_r,e_j,e_j,\partial_r).
\end{aligned}$$

Using the definition of the curvature endomorphism and its symmetries we see:

$$\begin{aligned}
Rm^{S^p\times D^q\times\mathbb{R}}(\partial_r,e_j,e_j,\partial_t) &= g\big(\underbrace{R(\partial_r,e_j,e_j)}_{\text{orthogonal to }\partial_t},\partial_t\big)=0\\
Rm^{S^p\times D^q\times\mathbb{R}}(\partial_t,e_j,e_j,\partial_r) &= Rm^{S^p\times D^q\times\mathbb{R}}(e_j,\partial_t,\partial_r,e_j)=Rm^{S^p\times D^q\times\mathbb{R}}(\partial_r,e_j,e_j,\partial_t)\\
Rm^{S^p\times D^q\times\mathbb{R}}(\partial_t,e_j,e_j,\partial_t) &= g(R(\partial_t,e_j,e_j),\partial_t)\\
R(e_j,\partial_t,\partial_t) &= \nabla_{e_j}\underbrace{\nabla_{\partial_t}\partial_t}_{=0}+\nabla_{\partial_t}\underbrace{\nabla_{e_j}\partial_t}_{=0}+\nabla_{\underbrace{[\partial_t,e_j]}_{=0}}\partial_t=0
\end{aligned}$$

Shuffling all these formulae together, we obtain:[1]

$$\begin{aligned}
\sum_{i<j}\lambda_i\lambda_j &= k\sum_{i=2}^{q}\lambda_i+k\sum_{i=q+1}^{n}\lambda_i+\sum_{2\le i<j\le q}\lambda_i\lambda_j+\sum_{2\le i\le q<j\le n}\lambda_i\lambda_j+\sum_{q<i<j\le n}\lambda_i\lambda_j\\
&= \sin\theta\cdot k\cdot(q-1)\cdot\big(-\frac{1}{r}+O(r)\big)+k\cdot O(1)\cdot\sin\theta\\
&\quad +\underbrace{\frac{(q-1)(q-2)}{2}}_{=\binom{q-1}{2}}\cdot\underbrace{\big(-\frac{1}{r}+O(r)\big)^2}_{=\left(\frac{1}{r^2}+O(1)\right)}\cdot\sin^2\theta\\
&\quad +(q-1)\cdot\big(-\frac{1}{r}+O(r)\big)\cdot O(1)\cdot\sin^2\theta+O(1)\cdot\sin^2\theta\\
&= -\sin\theta\cdot k\cdot\frac{q-1}{r}+\sin\theta\cdot k\cdot(q-1)\cdot O(r)+k\cdot\sin\theta\cdot O(1)\\
&\quad +\sin^2\theta\cdot\frac{(q-1)(q-2)}{2\cdot r^2}+\sin^2\theta\cdot\underbrace{\frac{(q-1)(q-2)}{2}\cdot O(1)}_{=O(1)}+\frac{q-1}{r}\cdot\sin^2\theta\cdot O(1)\\
&\quad +(q-1)\cdot\sin^2\theta\cdot O(r)+\sin^2\theta\cdot O(1)\\
&= \sin^2\theta\cdot O(1)+k\cdot\sin\theta\cdot\Big(-\frac{q-1}{r}+O(1)\Big)\\
&\quad +\sin^2\theta\cdot\Big(\frac{(q-1)(q-2)}{2\cdot r^2}+\frac{q-1}{r}\cdot O(1)\Big)\\
&\quad +\sin\theta\cdot k\cdot(q-1)\cdot O(r).
\end{aligned}$$

[1] This is the point, where the mistake in [14] occurs: $\binom{q-1}{2}$ is miscounted as $(q-1)(q-2)$.

And finally:

$$\begin{aligned}
\kappa^{M_\gamma} &= \sum_{i\neq j} Rm^{M\gamma}(e_i, e_j, e_j, e_i) = 2\sum_{i<j} Rm^{M\gamma}(e_i, e_j, e_j, e_i) \\
&= 2\sum_{i<j}\Big(Rm^{S^p\times D^q\times\mathbb{R}}(e_i, e_j, e_j, e_i) + \lambda_i\lambda_j\Big) \\
&= 2\sum_{i<j}\Big(Rm^{S^p\times D^q}(e_i, e_j, e_j, e_i) + \lambda_i\lambda_j\Big) + \sin^2\theta \cdot \underbrace{2\cdot\sum_{j=2}^{n} Rm^{S^p\times D^q}(\partial_r, e_j, e_j, \partial_r)}_{=2\cdot Ric^{S^p\times D^q}(\partial_r,\partial_r)=O(1)} \\
&= \kappa^{S^p\times D^q} + 2\sum_{i<j}\lambda_i\lambda_j + \sin^2\theta\cdot O(1) \\
&= \kappa^{S^p\times D^q} + \sin^2\theta\cdot O(1) + k\cdot\sin\theta\cdot\Big(-\frac{2\cdot(q-1)}{r} + O(1)\Big) \\
&\quad + \sin^2\theta\cdot\Big(\frac{(q-1)(q-2)}{r^2} + \frac{2\cdot(q-1)}{r}\cdot O(1)\Big) + \sin\theta\cdot k\cdot(q-1)\cdot O(r).
\end{aligned}$$

□

Remark 3.12. *The same proof also works if we take a closed submanifold N^p with trivial normal bundle instead of the embedded surgery sphere S^p. We then start with an embedding $N^p \times D^q \to M$ and derive the same curvature formula:*

$$\begin{aligned}
\kappa^{M_\gamma} &= \kappa^{N^p\times D^q} + \sin^2\theta\cdot O(1) + k\cdot\sin\theta\cdot\Big(-\frac{2\cdot(q-1)}{r} + O(1)\Big) \\
&\quad + \sin^2\theta\cdot\Big(\frac{(q-1)(q-2)}{r^2} + \frac{2\cdot(q-1)}{r}\cdot O(1)\Big) + \sin\theta\cdot k\cdot(q-1)\cdot O(r).
\end{aligned}$$

3.2.2 The initial bending

In the beginning we have $\sin\theta = 0$ and all terms except for $\kappa^{S^p\times D^q}$ in (3.8) vanish. Since M is a psc manifold, this is positive. We leave $r \equiv r_0$ for a short time and then perform a small bend of γ, which means, we choose k (the curvature of γ) as a continuous bump function in r.

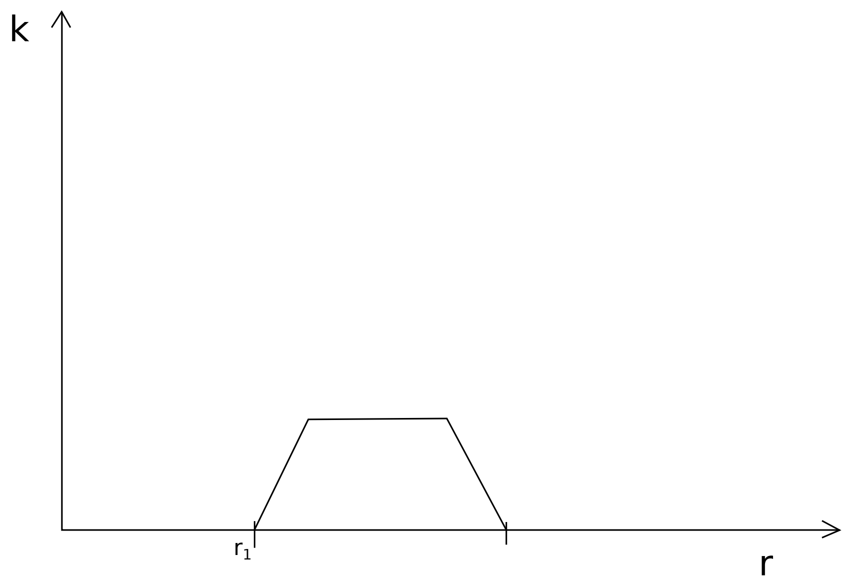


Figure 3.7: The bump function $k(r)$ for the initial bending

Since every term in the formula depends continuously on k ($\Delta\theta = \int kds$), we can maintain $\kappa^{M_\gamma} > 0$ (this is an open condition), if the bump function is small enough. After this small bending we take γ as a straight line ($\theta \equiv \theta_0$) until r is small enough, so that we can control the $O(1)$ and $O(r)$ terms in (3.8).

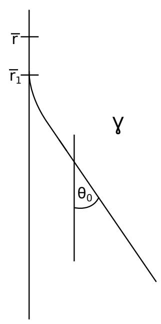


Figure 3.8: Result of the initial bending

3.2.3 The final bending

Let $\theta \geq \theta_0 > 0$. We want $\kappa^{M_\gamma} > 0$.

$$0 < K^{M_\gamma}$$

$$\begin{aligned}
\iff\ & 0 < \kappa^{S^p \times D^q} + \sin^2\theta \cdot O(1) + k \cdot \sin\theta \cdot \Big(-\frac{2\cdot(q-1)}{r} + O(1)\Big) \\
& \qquad + \sin^2\theta \cdot \Big(\frac{(q-1)(q-2)}{r^2} + \frac{2\cdot(q-1)}{r}\cdot O(1)\Big) + \sin\theta \cdot k \cdot (q-1)\cdot O(r) \\
\iff\ & k \cdot \sin\theta \cdot [\frac{2\cdot(q-1)}{r} + O(1) + (q-1)\cdot O(r)] \\
& \qquad < \kappa^{S^p \times D^q} + \sin^2\theta \cdot \Big(O(1) + \frac{(q-1)(q-2)}{r^2} + \frac{2\cdot(q-1)}{r}\cdot O(1)\Big) \\
\iff\ & k \cdot [2\cdot(q-1) + O(r) + (q-1)\cdot O(r^2)] \\
& \qquad < \frac{\kappa^{S^p \times D^q}\cdot r}{\sin\theta} + \sin\theta \cdot \Big(r\cdot O(1) + \frac{(q-1)(q-2)}{r} + 2\cdot(q-1)\cdot O(1)\Big) \\
\iff\ & k \cdot [1 + O(r) + O(r^2)] \\
& \qquad < \frac{\kappa^{S^p \times D^q}\cdot r}{2\cdot(q-1)\cdot\sin\theta} + \frac{\sin\theta\cdot r}{2(q-1)}\cdot O(1) + \sin\theta\cdot\Big(\frac{(q-2)}{2\cdot r} + O(1)\Big).
\end{aligned}$$

Let $C > 0$ be the constant from (3.11). Then:

$$\begin{aligned}
& k\cdot[1+O(r)+O(r^2)] && < \frac{\kappa^{S^p \times D^q}\cdot r}{2\cdot(q-1)\cdot\sin\theta} + \frac{\sin\theta\cdot r}{2(q-1)}\cdot O(1) + \sin\theta\cdot\Big(\frac{(q-2)}{2\cdot r} + O(1)\Big) \\
\Leftarrow\quad & k\cdot\underbrace{[1+C\cdot r + C\cdot r^2]}_{<\frac{3}{2}\text{ for } r \text{ small enough}} && < \underbrace{\frac{\kappa^{S^p \times D^q}\cdot r}{2\cdot(q-1)\cdot\sin\theta}}_{>0} - \frac{\sin\theta\cdot r}{2(q-1)}\cdot C + \sin\theta\cdot\Big(\frac{(q-2)}{2\cdot r} - C\Big) \\
\Leftarrow\quad & \tfrac{3}{2}\cdot k && \leq \frac{\sin\theta}{2\cdot r}\underbrace{\Big(\underbrace{-r^2\cdot\frac{C}{q-1} + (q-2) - C\cdot r^2}_{\to(q-2)\geq 1 \text{ for } r\to 0}\Big)}_{>\frac{3}{4}\text{ for } r \text{ small enough}} \\
\Leftarrow\quad & k && \leq \frac{\sin\theta}{4\cdot r}.
\end{aligned}$$

In other words: For some $r_0 > 0$ we have: If $r \leq r_0$ and $k = \frac{\sin\theta}{4\cdot r}$, then

$$\kappa^{M_\gamma} > 0.$$

Remark 3.13. *This is the point where we see that the mistake in the curvature formulae does have some significance. In the formulae of [6], [13] and [14], we either have* $k\cdot\sin\theta\cdot\frac{(q-1)}{r}$ *instead of* $k\cdot\sin\theta\cdot\frac{2\cdot(q-1)}{r}$ *or* $2\cdot\frac{(q-1)(q-2)}{r^2}$ *instead of* $\frac{(q-1)(q-2)}{r^2}$. *In both cases we can derive the condition* $k = \frac{\sin\theta}{2\cdot r}$ *for positive scalar curvature instead of* $k = \frac{\sin\theta}{4\cdot r}$. *If* $(q-2)$ *is at least* 2, *i.e.* q *is at least* 4, *we can get* $k = \frac{\sin\theta}{2\cdot r}$ *and we can follow the proof of [6] and [13]. However, if* $q = 3$, *we have to find another argument.*

In the end, γ should look like this:

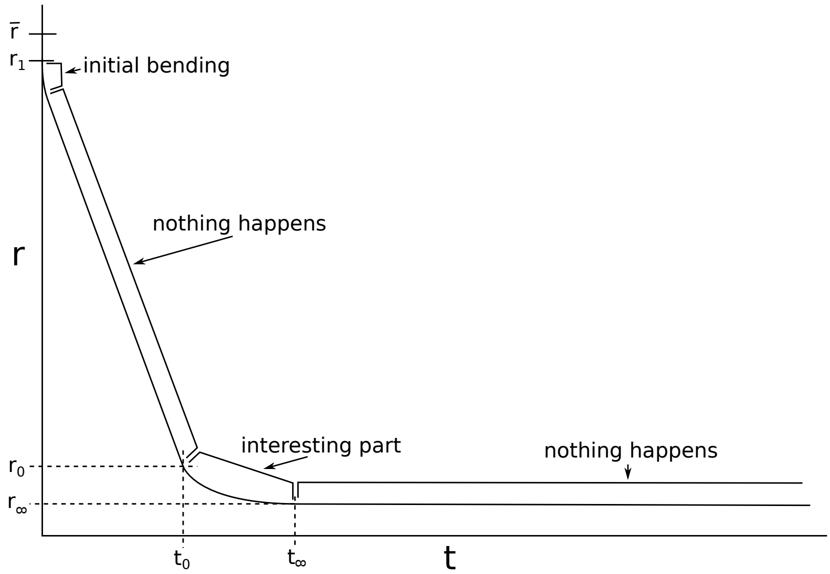


Figure 3.9: The graph of the bending map γ

In the interesting part (or even right after the initial bending), γ can be seen as a curve parametrized in t:

$$\gamma(t) = \begin{pmatrix} t \\ f(t) \end{pmatrix}$$

for some height function f. From [1, p. 41], we know that the curvature of γ is given by

$$\begin{aligned} k &= \frac{\det(\dot{\gamma}, \ddot{\gamma})}{\|\dot{\gamma}\|^3} = \frac{f''}{\left(\sqrt{1+f'^2}\right)^3} \\ \sin\theta &= \frac{1}{\|\dot{\gamma}\|} = \frac{1}{\sqrt{1+f'^2}}. \end{aligned}$$

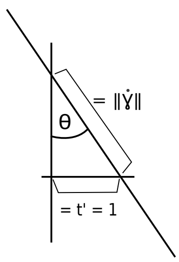


Figure 3.10: Computing the sine as $\frac{\text{opposite side}}{\text{hypotenuse}}$

Hence,

$$k = \frac{\sin\theta}{4\cdot r} \iff \frac{f''}{\left(\sqrt{1+f'^2}\right)^3} = \frac{1}{4\cdot f\cdot\sqrt{1+f'^2}} \iff f'' = \frac{1+f'^2}{4\cdot f}.$$

If this differential equation yields a solution which is nice enough, we are done with the proof of (3.1). This is ensured by the following lemma.

Lemma 3.14. *The second order ordinary differential equation*

$$f'' = \frac{1+f'^2}{a\cdot f} \tag{3.15}$$

has for $a > 0$ and starting values $f(t_0) > 0$ and $f'(t_0) < 0$ a solution, such that:

1. *f is defined on a closed interval $[t_0, T]$, $T > t_0$*
2. *f is strictly positive.*
3. *$f' \leq 0$ and $f'(t) = 0 \iff t = T$.*

Proof. A solution of the differential equation

$$f' = -\sqrt{f^{\frac{2}{a}}\cdot c - 1},$$

where c is a constant depending on the starting values will be a solution of equation (3.15). This can be seen by differentiating the equation from above:

$$\begin{aligned} f'' &= \frac{1}{-2\sqrt{f^{\frac{2}{a}}\cdot c - 1}}\cdot c\cdot\frac{2}{a}f^{\frac{2}{a}-1}\cdot f' = \frac{1}{a}\cdot c\cdot f^{\frac{2}{a}-1} \\ &= \frac{1}{a\cdot f}(\underbrace{f^{\frac{2}{a}}\cdot c - 1}_{=f'^2}+1) = \frac{1+f'^2}{a\cdot f}. \end{aligned}$$

From $f(t_0) > 0$ and $f'(t_0) < 0$ we know that c has to be greater then 0.

Furthermore, we see:

$$f' < 0 \iff f^{\frac{2}{a}} > \frac{1}{c} \iff f > \left(\frac{1}{c}\right)^{\frac{a}{2}} > 0 \tag{3.16}$$

$$f' = 0 \iff f^{\frac{2}{a}} = \frac{1}{c} \iff f = \left(\frac{1}{c}\right)^{\frac{a}{2}}. \tag{3.17}$$

As we are only interested in this solution as long as $f' \leq 0$, we know from (3.16 and 3.17) that in this case f is bounded from below by $\left(\frac{1}{c}\right)^{\frac{a}{2}} > 0$ and by $f(t_0)$ from above. Until now, we know that there is a solution to the differential equation which is strictly decreasing in the beginning. We only need to show that there is a point T, where $f'(T) = 0$.

f'' is given by

$$f'' = \frac{f^{\frac{2}{a}}\cdot c}{a\cdot f} = \frac{c}{a}\cdot f^{\frac{2}{a}-1} \geq \begin{cases} \frac{c}{a}\cdot B^{\frac{2}{a}-1} > 0 & \text{if } a\in(0,2] \\ \frac{c}{a}\cdot\left(\frac{1}{c}\right)^{\frac{2}{a}-1} > 0 & \text{if } a > 2 \end{cases} \tag{3.18}$$

and thus f'' is uniformly bounded from below.

If f' was strictly positive, f would be strictly decreasing for $t \to \infty$. Since f is bounded from below, it had to converge to some constant A. In this case, every derivative of f would converge to 0. This is a contradiction to the fact that f'' is bounded from below by some positive constant (see 3.18). Hence, there has to be some $T > t_0$, such that $f'(T) = 0$. This concludes the proof of (3.14). $\square$

Remark 3.19. *If $a = 2$ as in [6], [13] and [14], one can write down an explicit solution:*

$$f(t) = \frac{1}{A} + \frac{A}{4}\big(t - B\big)^2$$

for some constants $A, B > 0$.

4 The Gromov-Lawson-Chernysh construction

In this chapter, we are going to construct a map that deforms a compact family of metrics into a family of standard metrics. This result was first proven by Pawel Gajer in [5] for only 1 metric instead of a family of metrics.

Recall Lemma (2.66):

Lemma 2.66 ([6, p. 425], [14, p.73])**.** *We have:*

1. *The principal curvature of an embedded hypersurface $S^{n-1}(\varepsilon)$ are of the form $-\frac{1}{\varepsilon} + O(\varepsilon)$ for small epsilon.*
2. *The term $O(\varepsilon)$ depends on the metric and its derivatives.*
3. *Let g_ε be the induced on $S^{n-1}(\varepsilon)$ and let $g_{0,\varepsilon}$ be the standard euclidian metric of curvature $\frac{1}{\varepsilon^2}$. Then as $\varepsilon \to 0$, $\frac{1}{\varepsilon^2} g_\varepsilon \to \frac{1}{\varepsilon^2} g_{0,\varepsilon} = g_{0,1}$ in the C^2-topology described in (2.7).*

As an easy corollary, we get:

Corollary 4.1. *If $(g_s)_{s \in S}$ is a compact family of metrics, the constant C (3.11) in the proof of the* Gromov-Lawson *surgery theorem can be chosen, such that it works for all the metrics g_s.*

4.1 Admissible curves

Definition 4.2. Let $(g_s)_{s\in S}$ be a continuous family of metrics on M over a compact space S.
An admissible curve is a smooth curve $\gamma\colon \mathbb{R} \to \mathbb{R}^2 = \{(x,t)\}$ that satisfies the following conditions:

1. $\gamma(0) = (0, r_0)$ for an $r_0 > 0$ and $\gamma(t) = (0, r_0 - t)$ for all $t \leq 0$, hence $\dot{\gamma}(0) = (0, -1)$.
2. γ intersects the t-axis only once, at a right angle and follows the arc of a circle around this point.
3. γ is symmetric to the point $L \in \mathbb{R}$ where γ crosses the t-axis, i.e. $\gamma(L-s) = R_t \circ \gamma(L+s)$, where R_t is the reflection about the t-axis.
4. The injectivity radius of the exponential map of all metrics (g_s) is strictly greater than r_0.

The set of admissible curves with respect to the family S will be denoted by Γ_S and will be equipped with the C^∞-topology as a subspace of $C^\infty(\mathbb{R}, \mathbb{R}^2)$.

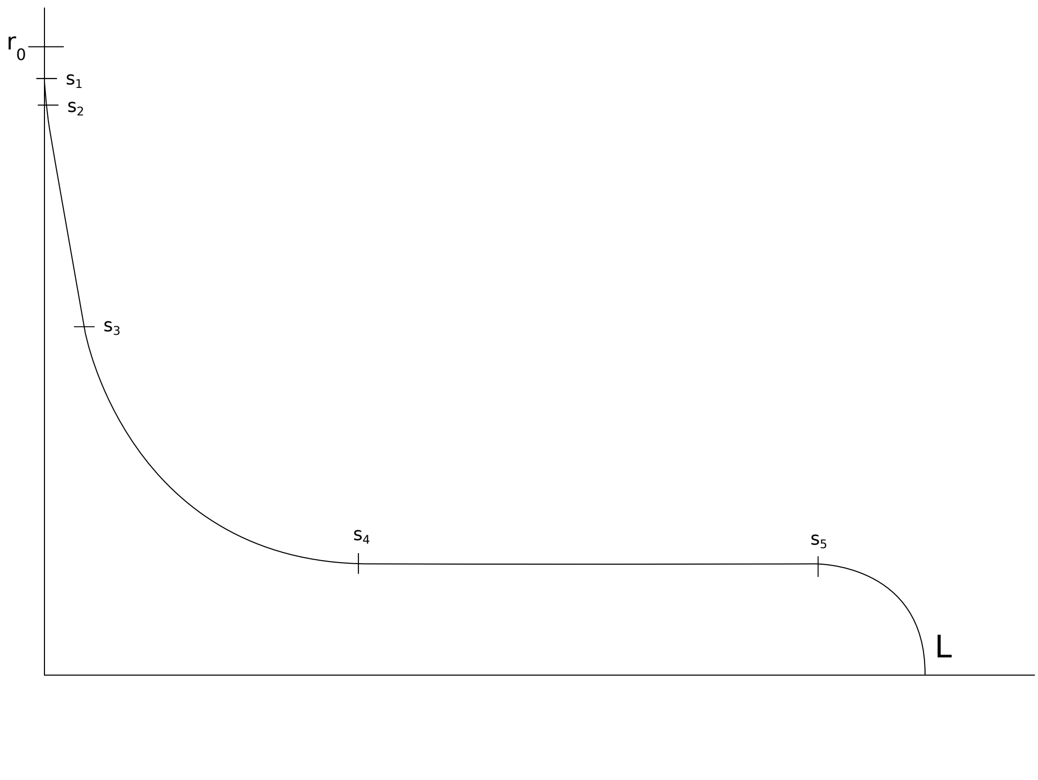


Figure 4.1: An admissble curve on $[0, L]$

Remark 4.3. *Any curve $\gamma \in \Gamma_S$ is uniquely determined by its part on $[0, L]$ and also by its curvature function on $[0, L]$.*
The r-axis is an admissible curve.

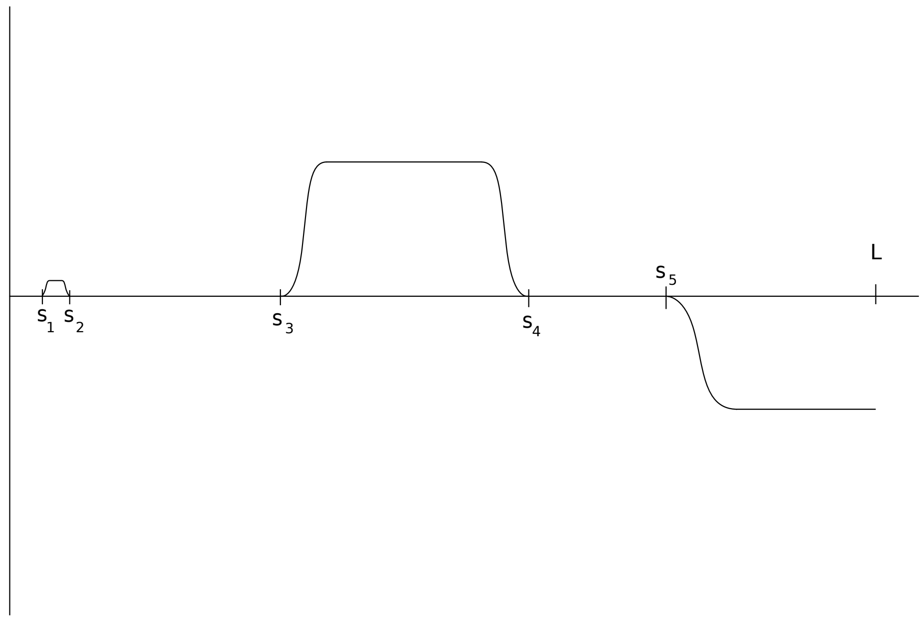


Figure 4.2: The curvature function of an admissible curve on $[0, L]$

Remark 4.4. *The curve γ from the proof of (3.1) is an admissable curve. For such a curve γ we divide the interval $[0, L]$ into 6 sub intervals:*

On $[0, s_1]$ the curve goes straight down.

On $[s_1, s_2]$ the initial bending takes place.

On $[s_2, s_3]$ the curve follows a straight line.

On $[s_3, s_4]$ is the part where the upwards beding is performed.

On $[s_4, s_5]$ the curve follows a straight horizontal line.

On $[s_5, L]$ it is bent downwards and finally intersects the t-axis.

The division into these intervals can be seen in (Figure 4.1 & Figure 4.2). We call an admissible curve that originates from the proof of (3.1) a GROMOV-LAWSON *curve or a* **GL**-*curve.*

Proposition 4.5. *The map*

$$\begin{array}{rcl} C^\infty(\mathbb{R},\mathbb{R}) & \to & C^\infty(\mathbb{R},\mathbb{R}^2) \\ k & \mapsto & \gamma(k), \end{array}$$

which sends a curvature function $k(s)$ to the admissible curve corresponding to it, is continuous.

Proof. $\gamma(k)$ can be defined as the unique solution of the second order differential equation:

$$\begin{array}{rcl} \gamma(0) & = & (0, r_0) \\ \dot\gamma(0) & = & (0, -1) \\ \ddot\gamma(s) & = & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \dot\gamma(s) \cdot k(s). \end{array}$$

By (2.57) this map is continuous. As any admissible curve is uniquely determined by its part on $[0, L]$ this completes the proof. $\square$

Definition 4.6. The Neck $T_\gamma(s) \subset M \times \mathbb{R}$ of an admissible curve near some closed submanifold N with trivial normal bundle is defined as:

$$T_\gamma(g_s) = \{(x, y, t) \in N \times D^q \times \mathbb{R} : \gamma(t) = \|y\|_{g_s}\}.$$

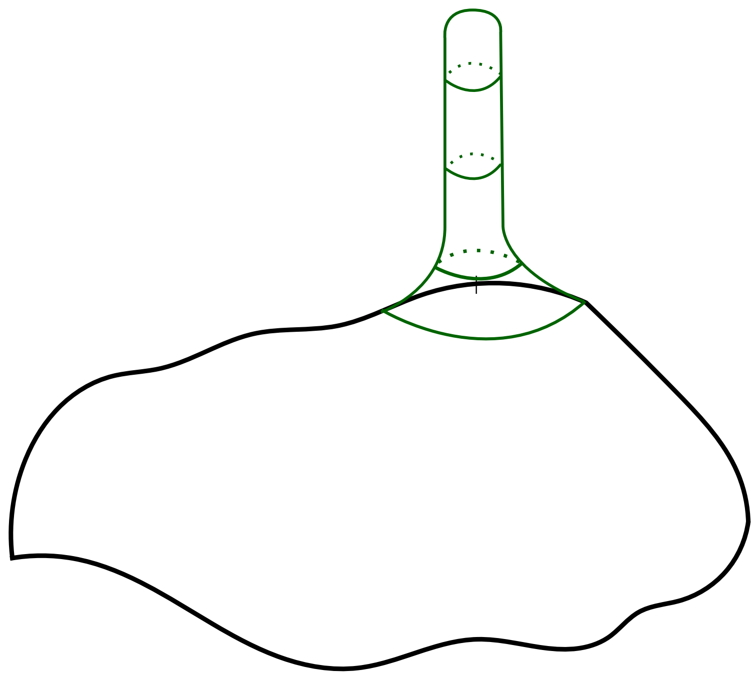


Figure 4.3: The neck of an admissible curve for $N = *$

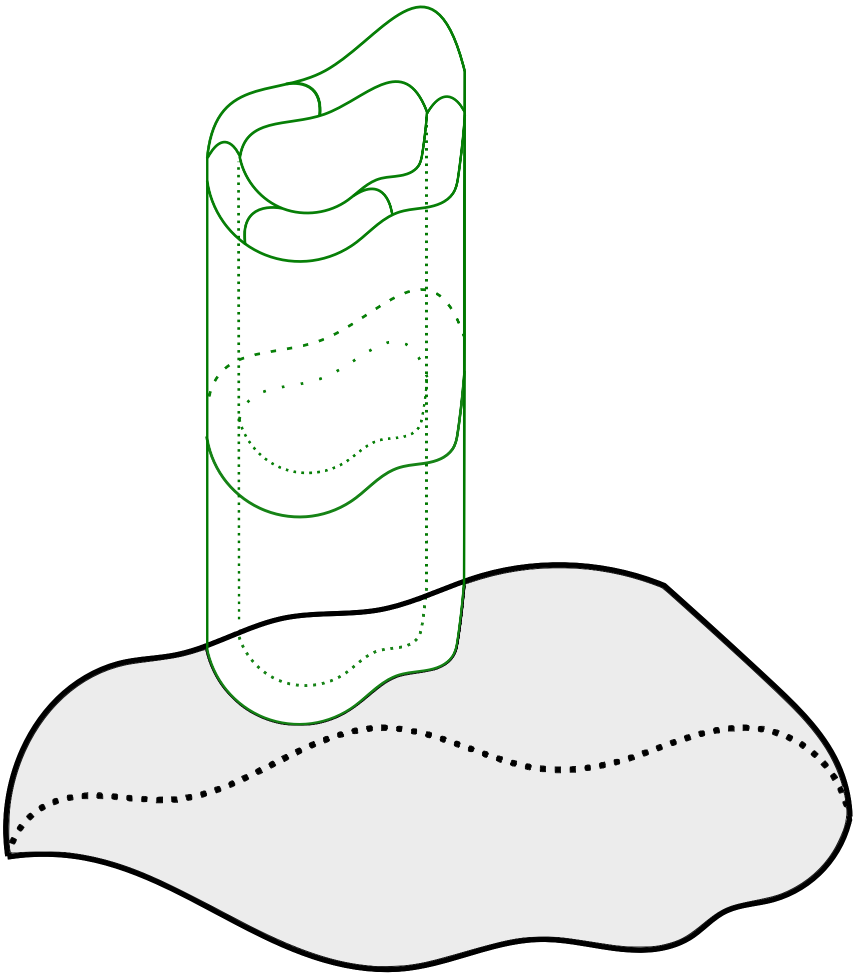


Figure 4.4: The neck of an admissible curve for an embedded $N = S^1$.

Proposition 4.7 ([3, pp. 3-4])**.** *Let $(g_s)_{s \in S}$ be a compact family of Riemannian metrics on M.*
Then there exists a continuous map

$$\Gamma_S \times S \to Emb(M, M \times \mathbb{R}),$$

where $Emb(M, M \times \mathbb{R}) \subset C^\infty(M, M \times \mathbb{R})$ is equipped with the subspace topology. This map satisfies the following properties:

1. *The resulting embedding $f_{\gamma,s}$ is constant outside the tubular neighbourhood of radius r_0 around N, which will be denoted by $Tb_{r_0}(N)$, i.e. $f(x) = (x, 0)$ on $Tb_{r_0}(N)$.*

2. *$f_{\gamma,s}$ diffeomorphically maps $Tb_{r_0}(N)$ onto the neck of $T_\gamma(g_s)$, which is the same as $Tb_L(N_\gamma(g_s))$, where $N_\gamma(g_s)$ is the submanifold of $T_\gamma(g_s)$ which is diffeomorphic to N under the projection map $M \times \mathbb{R} \to M$.*

Proof. Let $L > 0$ be the length of the admissible curve $\gamma \in \Gamma_S$ and let ϕ be a smooth, strictly increasing function depending continuously on r_0 and L:

$$\phi(r) = \begin{cases} r & r \leq \delta \\ r + r_0 - L & r \geq L \end{cases}$$

for some $0 \leq \delta < r_0$.

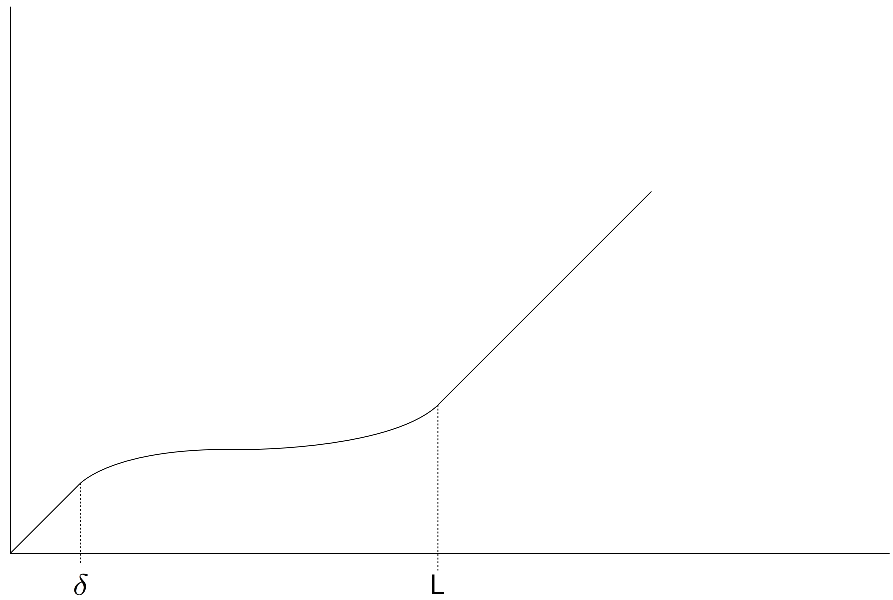


Figure 4.5: The function ϕ

The map ϕ gives rise to a rescaling diffeomorphism $\phi\colon \nu_M^N \to \nu_M^N, (x, v) \mapsto (x, \frac{\phi(\|v\|)}{\|v\|} \cdot v)$. Let N_γ be the submanifold of $T_\gamma(g_s)$ which is diffeomorphic to N under the projection map $p\colon M \times \mathbb{R} \to M$, i.e. $N_\gamma = p^{-1}(N) \cap T_\gamma$. The composition of maps

$$Tb_L(N_\gamma) \xrightarrow{(\exp^\perp_{T_\gamma(g_s)})^{-1}} \left(\nu^{N_\gamma(g_s)}_{M_\gamma(g_s)}\right)_L \xrightarrow{dp} \left(\nu_M^N\right)_L \xrightarrow{\phi} \left(\nu_M^N\right)_{r_0} \xrightarrow{\exp^\perp_{g_s}} Tb_{r_0}(N)$$

then defines a diffeomorphism by (2.26), which maps points x that have distance equal to r_0 to N to $(x, 0)$. Here $\left(\nu_M^N\right)_L$ denotes all vectors of length at most L in $\left(\nu_M^N\right)$. By mapping all points outside of $Tb_{r_0}(N)$ to $(x, 0)$, we get the required embedding. It remains to be shown that the map $(\gamma, g_s) \mapsto f_{\gamma, g_s} := \exp^\perp_{T_\gamma(g_s)} \circ dp^{-1}_{\gamma, g_s} \circ \phi^{-1} \circ (\exp^\perp_{g_s})^{-1}$ is continuous.

By [9, p. 58 & p. 72] the map $\exp(V) = \alpha_V(1)$ is defined as the solution of the differential equation

$$\begin{aligned} \alpha(0) &= \pi(V) \\ \dot\alpha(0) &= V \\ D_t\alpha &\equiv 0, \text{ which can be expressed in local coordinates:} \\ & \ddot{x}^k(t) + \dot{x}^i(t)\dot{x}^j(t)\Gamma^k_{ij}(x(t)) = 0. \end{aligned}$$

Since the Christoffelsymbols depend continuously on the metric and its first derivative we derive continuity of $\gamma, g_s \mapsto \phi^{-1} \circ (\exp^{\perp}_{g_s})^{-1}$ by (2.57).

Let's examine dp^{-1} next. Let $\big((x,t),v\big) \in \nu^{N_\gamma}_{M_\gamma}$. Then $dp\big((x,t),v\big) = (x,v)$, as any admissible curve ends with a vertical segment and thus a vector normal to N_γ is parallel to M and can be interpreted as a normal vector of N in M (see (Figure 4.6)).

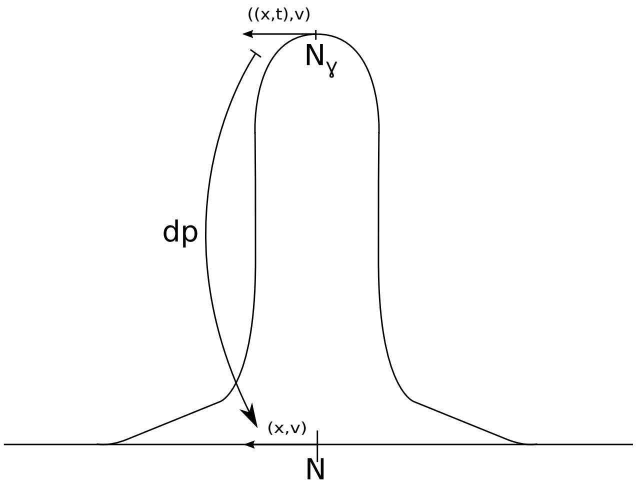


Figure 4.6: The differential of the projection map

So we get that $dp_\gamma^{-1}(x,v) = \big((x,t),v\big)$, where t is the unique real number, such that $(x,t) \in N_\gamma$. If (γ_n) is a sequence converging to γ in C^∞, we get that t_n converges to t. For the derivative of dp_γ^{-1} we notice:

$$\begin{aligned} p^{-1}\colon M &\to M \times \mathbb{R} \\ x &\mapsto (x,t) \\ dp^{-1}\colon TM &\to TM \times T\mathbb{R} \\ (x,v) &\mapsto \big((x,v),(t,0)\big) \\ ddp^{-1}\colon TTM &\to TTM \times TT\mathbb{R} \\ \big((x,v),w\big) &\mapsto \big(((x,v),w),(t,0,0)\big). \end{aligned}$$

Analogously, one can compute any derivative of dp^{-1} and, since $t_n \to t$ for all these maps, $dp^{-1}_{\gamma_n} \to dp^{-1}_\gamma$ in the C^∞-topology.

The last map we have, is again an exponential map, which is defined as the solution of an ordinary differential equation:

$$\alpha(0) = \pi(v,0),\ \dot{\alpha}(0) = (v,0),\ \tilde{D}_t\dot{\alpha} \equiv 0.$$

The Christoffel symbols defining $\tilde{D}_t\dot{\alpha}$ come from the metric $(g_s + dt^2)|_{M_{\gamma,g_s}}$, depending continuously on g_s. Again by (2.57) we derive continuity of this map and whence we get that the constructed map is continuous. □

Corollary 4.8. *Let $(g_s)_{s\in S}$ be a compact family of metrics and let $N^p \subset M$ be a closed submanifold of codimension at least 3. Then there exists a* **GL**-*curve $\gamma \in \Gamma_S$, such that $T_\gamma(g_s)$ has positive scalar curvature for alle $s \in S$. Furthermore, we can choose γ, such that $k \le \frac{sin(\theta)}{4\cdot r}$ on $[s_3, s_4]$ implies positive scalar curvature on $T_\gamma(g_s)$ and from (3.14) we know that we can choose the curve γ, such that $k \le \frac{sin(\theta)}{5\cdot r}$ on $[s_3, s_4]$ will be satisfied.*

Proof. This is a direct consequence of (2.66), (4.1) and the original Gromov-Lawson construction (3.1). □

Definition 4.9. A δ-cutoff function $\delta_{s_0}(s)$ at a point $s_0 \in [0, L]$ is a smooth, strictly decreasing map $[0, L] \to [0, 1]$, such that $\delta_{s_0}|_{[0,s_0]} = 1$ and $\delta_{s_0}|_{[s_0+\delta,L]} = 0$.

Proposition 4.10 ([3, p. 5]])**.** *Let $\gamma \in \Gamma_S$ be a* **GL**-*curve with $k \le \frac{sin(\theta)}{5\cdot r}$ on $[s_3, s_4]$. Then there is a $\delta > 0$, such that for all $s_0, s \in [s_3, s_4]$ and for $\tilde{\gamma} = \gamma(\tilde{k}) = \gamma(\delta_{s_0}\cdot k)$ the following inequality holds:*

$$\tilde{k}(s) := \delta_{s_0}(s)k(s) \le \frac{sin(\tilde{\theta}(s))}{4\cdot \tilde{r}(s)}.$$

Here $\tilde{\theta}$ and $\tilde{r}$ belong to the curve $\tilde{\gamma}$.

Proof. Consider the function

$$\begin{aligned} F\colon [s_3,s_4]\times[0,1] &\to \mathbb{R} \\ (s,t) &\mapsto k(s+t) - \frac{\sin(\theta(s))}{4\cdot r(s)}. \end{aligned}$$

Then, $F(s,0) = k(s) - \frac{\sin(\theta(s))}{4\cdot r(s)} \le A_0 < 0$ for some $A_0 < 0$ and so there exists a $\delta > 0$, such that $F([s_3, s_4]\times[0,\delta]) \subset (-\infty, A_1)$ for some $A_1 < 0$. Let δ_{s_0} be a cutoff function. Then we have for all $s \in [0,\delta]$:

$$k(s+s_0) < \frac{\sin(\theta(s))}{4\cdot r(s)}$$

and thus

$$\begin{aligned}\tilde{k}(s_0+s) &= \delta_{s_0}k(s_0+s)\\ &\le k(s_0+s) < \frac{\sin(\theta(s_0))}{4\cdot r(s_0)} = \frac{\sin(\overbrace{\tilde{\theta}(s_0)}^{\text{increasing}})}{4\cdot \underbrace{\tilde{r}(s_0)}_{\text{decreasing}}}\\ &\le \frac{\sin(\tilde{\theta}(s_0+s))}{4\cdot\tilde{r}(s_0+s)}\end{aligned}$$

and therefore

$$\tilde{k}(s) \le \frac{\sin\tilde{\theta}(s)}{4\cdot\tilde{r}(s)}$$

for all $s \in [s_3, s_4]$. □

Theorem 4.11 ([3, p. 6]). *Let $(g_s)_{s\in S}$ be a compact family of metrics. Let $\gamma \in \Gamma_S$ be a* **GL***-curve from (4.8) that satisfies $k \le \frac{sin(\theta)}{5\cdot r}$ on $[s_3, s_4]$. Then there is a continuous map $\alpha_1 \colon I \to \Gamma_S$, such that*

1. $\alpha_1(0) = \gamma$
2. $\alpha_1(1) = \{r\text{-}axis\}$
3. $T_{\alpha_1(t)}(g_s)$ *has positive scalar curvature for every $s \in S, t \in I$.*

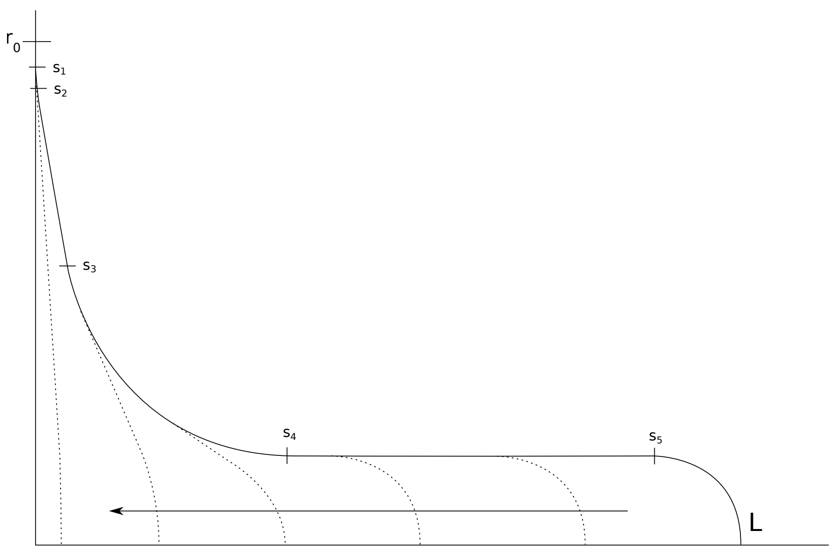


Figure 4.7: Pulling back of an admissible curve

Proof. Instead of deforming the curve itself, we deform its curvature function $k \in C^\infty([0,L],\mathbb{R})$ to $k \equiv 0$. We will perform the deformation, such that the first and second bend of γ stay at the same place.

Let $\delta < \min\{s_1, s_5 - s_4\}$. We define $p(t) := s_4 - s_4 \cdot t$ and take a family of δ-cutoff functions $\delta_{p(t)}$, such that $\delta_{p(t)+t_0}(s) = \delta_{p(t)}(s - t_0)$, i.e. we pick one δ-cutoff function and shift it for every s_0. Then $\delta_{p(t_n)}(s) = \delta_{p(t)+p(t_n)-p(t)}(s) = \delta_{p(t)}(s - p(t_n) + p(t))$ converges in the C^∞-topology to $\delta_{p(t)}$.

We then take $k_t = \tilde{k}_{p(t)}$ on $[0, p(t)+\delta]$ and leave $k \equiv 0$ until the point s_t, where the corresponding curve γ_t crosses the $r = r_\infty$ line. On the last part $[s_t, L_t]$ we take $k_t(s) = \delta_{s_t}(s_t + \delta - s)\frac{1}{\varepsilon_t}$, where ε_t and L_t are the unique numbers, such that

$$\begin{aligned} \int_{[s_t,L_t]} k_t(s)ds &= -\int_{[0,s_t]} k_t(s)ds \\ \big(\gamma(k_t)\big)(L_t) &\in t\text{-axis}. \end{aligned}$$

We now need to show that this map is continuous in t. First we note that $p(t), s_t, L_t$ and ε_t depend continuously on t. Now let (t_n) be a sequence converging to t. By (2.52) we can choose the identity as charts. We divide this proof into 3 cases:

<u>Case 1:</u> $x \in [0, p(t)+\delta]$

From the product rule, we know:

$$\begin{aligned} k_{t_n}^{(m)}(s) &= \big(\delta_{p(t_n)}(s)k(s)\big)^{(m)} = \sum_{l=0}^{m} \binom{m}{l} \underbrace{\delta_{p(t_n)}^{(l)}(s)}_{\to \delta_{p(t)}^{(l)}(s)} k^{(m-l)}(s) \\ &\to \sum_{l=0}^{m} \binom{m}{l} \delta_{p(t)}^{(l)}(s)k^{(m-l)}(s) = \big(\delta_{p(t)}(s)k(s)\big)^{(m)} = k^{(m)}(s). \end{aligned}$$

<u>Case 2:</u> $x \in [p(t)+\delta, s_t]$

Since $s_{t_n} \to s_t$ we get that $k_{t_n}^{(m)}(s) \to 0$ for every $m \geq 0$.

<u>Case 3:</u> $x \in [s_t, L_t]$

This also follows from convergence of the parameters and the fact that the parameters and the δ-cutoff functions depend continuously on t.

The last thing to be verified is positive scalar curvature:

On $[s_1, s_2]$ we only make the small bump function smaller, so the scalar curvature remains positive.

On $[s_2, s_3]$ and on $[s_4, s_5]$ we don't change anything.

On $[s_3, s_t]$ we have positive scalar curvature due to (4.10).

On $[s_t, L_t]$ we are performing a downwards bend, which has positive scalar curvature, as it always satisfies $k \leq 0 \leq \frac{\sin(\theta)}{4 \cdot r}$ and takes place below $r = r_\infty$. This implies positive scalar curvature. $\square$

Our goal is to deform a compact family of metrics into standard metrics. The first deformation will be taking a compact family $(g_s)_{s\in S}$ and pushing a small tubular neigbourhood of N out by going backwards through the deformation from (4.11). This will allows us to deform the metric on M by deforming the metric on the neck of γ:

$$\begin{array}{ccccc} S \times I & \xrightarrow{(4.11)} & S \times \Gamma_S & \xrightarrow{(4.7)} & S \times Emb(M, M \times \mathbb{R}) \\ (s,t) & \longmapsto & (s, \alpha_1(1-t)) & \longmapsto & (s, f_{\alpha_1(1-t),s}). \end{array}$$

4.2 Deformation into standard metrics

In this section we deform the metric on the neck to one that has standard form on a small tubular neighbourhood of N. For the rest of this chapter let g_N be a fixed metric on N. This does not need to be a psc metric. Furthermore, let g_0 be a torpedo metric on D^q of radius T_0, where T_0 can be chosen to be arbitrarily small but then remains fixed throughout this chapter. Without loss of generality we may assume that $g_N + g_0$ has positive scalar curvature.

Corollary 4.12. *Let g be a metric on M. Then for ε small enough there is a homotopy from g restricted to $N^p \times D^q(T_0)$ to the metric $h := g_N + g_0$ through psc metrics. This homotopy depends continuously on the given metric g.*

Proof. We only have to show the continuity. The rest follows from (3.5).
Since the homotopy in (3.5) originates from (2.65), it is given by a linear path. So it suffices to show that $(g,t) \mapsto (1-t)g + th$ is continuous in t and g, which follows immediately from the triangular inequality: Let $(g_n, t_n) \to (g,t)$. Then:

$$\begin{aligned} \|(1-t)g + th, (1-t_n)g_n + t_n h\|^k_{\psi,\varphi} &\leq \|(1-t)g + th, (1-t_n)g + t_n h\|^k_{\psi,\varphi} \\ &\quad + \|(1-t_n)g + t_n h, (1-t_n)g_n + t_n h\|^k_{\psi,\varphi} \\ &\leq \underbrace{\|(1-t)g + th, (1-t_n)g + t_n h\|^k_{\psi,\varphi}}_{\to 0 \text{ by } (2.64)} \\ &\quad + |1-t_n| \underbrace{\|g, g_n\|^k_{\psi,\varphi}}_{\to 0} + \underbrace{\|t_n h, t_n h\|^k_{\psi,\varphi}}_{=0} \\ &\to 0. \end{aligned}$$

□

Theorem 4.13 ([3, p. 7])**.** *Let $(g_s)_{s\in S}$ be a compact family of metrics and let γ be a* **GL**-*curve which originates from (4.8). Then there exists a continuous map*

$$\alpha_2 \colon I \times S \to \mathcal{R}(M \times \mathbb{R}),$$

such that:

1. $\alpha_2(s,0) = g_s + dt^2$
2. $\alpha_2(s,\lambda)|_{M_\gamma}$ *is a psc metric*

3. *$\alpha_2(s,t) = g_s + dt^2$ on a neighbourhood of M in $M \times \mathbb{R}$*

4. *$\alpha_2(s,1)|_{M_\gamma} = g_N + g_0$ on a neighbourhood of N_γ.*

Proof. We want to deform the metric on $T_\gamma(g_s)$, such that it stays the same near M. We divide $T_\gamma(g_s)$ into 3 parts:

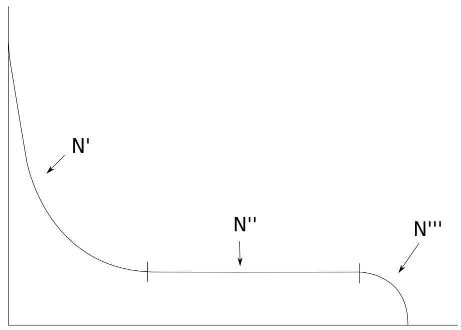


Figure 4.8: The admissible curve γ

On N' we choose the homotopy to be constant.

On N''' we take the homotopy G from $(g_s + dt^2)|_{N\times D^q(\varepsilon)}$ to $g_N + g_0$ (4.12). On the boundary $\partial(N \times D^q(\varepsilon)) = N \times S^{q-1}(\varepsilon)$ this gives a homotopy from g_s to the restriction of $g_N + g_0$.

On N'' we perform the deformation as follows: Let

$$h_\lambda(s) = G_\lambda(s)|_{N\times S^{q-1}}$$

be the restriction of the above homotopy. By (2.68) we get that there is an $a > 0$ such that the metric

$$h_{t/a}(s) + dt^2$$

on N'' is a psc metric. For $(x, (t/a)) \in S^{q-1}(\varepsilon) \times [0, 1]$ we get a homotopy

$$H\colon (\lambda, s) \mapsto h_{(\lambda t)/a}(s) + dt^2.$$

Then $H(0, s) = g_s + dt^2$ and $H(1, s) = h_{t/a}(s) + dt^2$ and for $t = a$ we have $H(\lambda, s) = h_\lambda + dt^2$, thus $H(\lambda, s) = h_\lambda(s)$ on the boundary of N'' towards N'''.

The deformation can now be written as:

$$\alpha_2(t,s)|_{M_\gamma} = \begin{cases} g_s + dt^2 & \text{on } N' \\ H(t,s) & \text{on } N'' \\ G(t,s) & \text{on } N'''. \end{cases}$$

We obtain a metric on $M \times \mathbb{R}$ by extending it arbitrarily.

Furthermore, on N''' we have $\alpha(1, s) = g_N + g_0$. $\square$

We can now extend the diagram from (4.1):

$$\begin{array}{ccccc} I \times S \times Emb(M, M \times \mathbb{R}) & \xrightarrow{(4.13)} & S \times \mathcal{R}(M \times \mathbb{R}) \times Emb(M, M \times \mathbb{R}) & \xrightarrow{(2.55)} & S \times \mathcal{R}^+(M) \\ (s, t, f_{\gamma,s}) & \longmapsto & \big(s, \alpha_2(t,s), f_{\gamma,s}\big) & \longmapsto & \big(s, (f_{\gamma,s})^* \alpha_2(t,s)\big) \end{array}$$

Since $N''' \cong N \times D^q$, we get that $\tau_s = (f_{\gamma,s})^{-1}$ is a tubular neighborhood that depends continuously on s. So we constructed a deformation:

$$\alpha_{1,2} \colon I \times S \to \mathcal{R}^+(M),$$

such that for every s there is a tubular neighbourhood τ_s depending continuously on s, satisfying

$$\tau_s^*\big(\alpha_{1,2}(1,s)\big) = g_N + g_0. \tag{4.14}$$

4.3 Final deformation

So far we we deformed a family of metrics $(g_s)_{s \in S}$ in a way that any resulting metric g_s satisfies (4.14). All the τ_s are given by $(f_{\gamma,s})^{-1} = \exp^{\perp}_{T_\gamma(g_s)} \circ dp^{-1}_{\gamma,g_s} \circ \phi^{-1} \circ (\exp^{\perp}_{g_s})^{-1}$. By the above computation (4.6) the differential of these maps at N is the same for all s. Let τ be a tubular neighbourhood of N which satisfies $D_0(\tau^{-1} \circ \tau_s) = id$ on the D^q part. We will deform the metrics, such that all of them have the standard form $g_N + g_0$ in the tubular neighbourhood τ. For this, we will reexamine the proof of uniqueness of tubular neigbourhoods.

Theorem 4.15. *Let $(g_s)_{s \in S}$ be a compact family of metrics and τ_s be a family of tubular neighbourhoods, depending continuously on s satisfying $\tau_s^* g_s = g_N + g_0$. Then there is a continuous map*

$$\alpha_3 \colon I \times S \to \mathcal{R}^+(M),$$

such that $\tau^*(\alpha_3(1,s)) = g_N + g_0$ for all $s \in S$.

Proof. First we fix a radial diffeomorphism $\Phi \colon D^q \cong \mathbb{R}^q$, which is the identity in a small neighbourhood of the origin. Instead of giving the homotopy of $\mathcal{R}^+(M)$, we give a diffeotopy of M

$$I \times S \times M \to M,$$

such that $H(s,0) = id$ and $\tau \circ H(s,1) = \tau_s$.

The first step is to reduce to the situation $\tau_s(N \times D^q) \subset \tau(N \times D^q)$. This is achieved by a applying a straight line homotopy of D^q: $(t,v) \mapsto v + t(\varepsilon v - v)$ for some small enough ε.

Since $\tau_s(N \times D^q) \subset \tau(N \times D^q)$ and since τ, τ_s are embeddings, we can consider the smooth map

$$h_s := \Phi \circ \tau^{-1} \circ \tau_s \circ \Phi^{-1} \colon N \times \mathbb{R}^q \to N \times \mathbb{R}^q.$$

This depends continuously on the metric g_s. We can now define a diffeotopy:

$$\begin{array}{rcl} H_s\colon I \times N \times \mathbb{R}^q & \to & N \times \mathbb{R}^q \\ (t,v) & \mapsto & H_s(t,v) = \begin{cases} \frac{1}{t} h_s(tv) & t > 0 \\ D_0 h_s \cdot v = v & t = 0. \end{cases} \end{array}$$

The map H_s is smooth in v and continuous in t and s. Furthermore we have

$$H_s(0, \_) = D_0 h_s = D_0 \Phi \circ D_0(\tau^{-1} \circ \tau_s) \circ D_0 \Phi^{-1} = D_0 \Phi \circ D_0 \Phi^{-1} = id$$

and

$$\Phi^{-1} \circ H_s(1, \Phi(\_)) = \tau^{-1} \circ \tau_s.$$

So we get an isotopy of tubular neighbourhoods by

$$\begin{array}{ccccccc} S \times I \times N \times D^q & \longrightarrow & S \times I \times N \times \mathbb{R}^q & \longrightarrow & N \times \mathbb{R}^q & \longrightarrow & N \times D^q \\ & & & & & & \\ (s,t,x,v) & \longmapsto & (s,t,x,\Phi(v)) & \longmapsto & \big(H_s(t,\Phi(v))\big) & \longmapsto & \big(\Phi^{-1} \circ H_s(t,\Phi(v))\big). \end{array}$$

By applying τ we can transport this isotopy to M and we get an isotopy $\tilde{H}_s(t, \_) = \tau \circ H_s(t, \_) \circ \tau^{-1}$ of $\tau(N \times D^q)$ into itself. We have the following equations:

$$\begin{array}{rcl} \tau_s^* \tilde{H}_s(0, \_)^* g_s & = & \tau_s^* g_s \\ \tau^* \tilde{H}_s(1, \_)^* g_s & = & \tau^*(\tau \circ H_s(1, \_) \circ \tau^{-1})^* g_s = \tau^*(\tau \circ \tau^{-1} \circ \tau_s \circ \tau^{-1})^* g_s \\ & = & \tau_s^* g_s = g_N + g_0. \end{array}$$

By the isotopy extension theorem [8, p. 181] we get a family of diffeotopies $G_s\colon I \times M \to M$ depending continuously on s, which are constant outside of a small neighbourhood of $\tau(N \times D^q)$ and agree on $\tau(N \times D^q)$ with $H_s(t, \_)$. Our homotopy of metrics then is

$$\begin{array}{rcl} S \times I & \to & \mathcal{R}^+(M) \\ (s,t) & \mapsto & G_s(t, \_)^* g_s, \end{array}$$

which satisfies

$$\begin{array}{rcl} G_s(0, \_)^* g_s & = & g_s \\ \tau^* G_s(1, \_)^* g_s & = & \tau_s^* g_s = g_N + g_0. \end{array}$$

This is continuous by (2.55). □

4.4 Putting the pieces together

In the last part of this chapter we will simply put the three homotopies together.

Theorem 4.16. *Let $q \geq 3$ and let $\tau\colon N^p \times D^q \hookrightarrow M^n$ be a tubular neighbourhood of N in M. Let g_0 be a torpedo metric of radius T_0, let g_N be a metric on N and let $(g_s)_{s \in S}$ be a compact family of psc metrics. Then there is a homotopy*

$$\mathbf{GLC}\colon I \times S \to \mathcal{R}^+(M),$$

such that

$$\begin{aligned} \mathbf{GLC}(0,s) &= g_s \\ \tau^*\mathbf{GLC}(1,s) &= g_N + g_0. \end{aligned}$$

This can be paraphrased to

$$\mathbf{GLC}(1,s) \in \mathcal{R}_0^+(M).$$

We call such a homotopy a Gromov-Lawson-Chernysh *deformation or a* **GLC**-*deformation for the family* $(g_s)_{s\in S}$.

Let $f_{\gamma,s}$ *be the embedding from (4.7). The homotopy* **GLC** *then can be written as:*

$$\mathbf{GLC}(\lambda,s) = \begin{cases} f^*_{\alpha_1(1-3\lambda),s}\big(g_s + dt^2\big) & \text{if } \lambda \in [0,1/3] \\ f^*_{\alpha_1(0),s}\big(\alpha_2(3\lambda-1,s)\big) & \text{if } \lambda \in [1/3,2/3] \\ \alpha_3(3\lambda-2,s) & \text{if } \lambda \in [2/3,1]. \end{cases}$$

Remark 4.17. *This deformation consists of first pushing out the neck belonging to an admissible curve* γ *(4.11), then deforming the metric on the neck into a standard form (4.13) and then pulling it back to* M *(4.7). Last but not least we use the unique tubular neighbourhood theorem and the isotopy extension theorem to make sure that all of these metrics have a standard form inside a fixed tubular neighbourhood (4.15).*

5 Warped metrics in the disk

The problem associated with the GROMOV-LAWSON-CHERNYSH-deformation is that it doesn't preserve the property of a metric being standard during the deformation. For this reason, we have to examine what happens inside the tubular neighbourhood τ, which means examining metrics on the disc. We start this chapter with the definition of a warped metric.

In the entire chapter T_0 shall be the radius belonging to a fixed torpedo metric g_0.

5.1 Scalar curvature of a warped product metric

Definition 5.1. Let $B \geq 0$. We define

$$W_B := \{h \in \mathcal{R}^+(D^q_{T_0}) \colon h = g(t)^2 dt^2 + f(t)^2 d\xi^2, g(t) \neq 0, \kappa > B\},$$

where $d\xi^2$ denotes the standard round metric on the sphere of radius 1. We call a metric of the form $h = g^2 dt^2 + f^2 d\xi^2$ a warped metric in the disc. Without loss of generality, we may assume $g > 0$ and $f \geq 0$.

Since $g > 0$, the map

$$\varphi \colon x \mapsto \frac{G^{-1}(\|x\|)}{\|x\|} x,$$

where

$$G(t) = \int_0^t g(s) ds$$

is a radial diffeomorphism. Here $\|.\|$ denotes the euclidian norm. Then the metric $\varphi^* h = \tilde{h}$ can be computed as follows: Let X, Y be tangent vectors at a point $t, s \in [0, T_0] \times S^{q-1}$. Then

$$\begin{aligned} \tilde{h}(X,Y) &= \varphi^* h(X,Y) = h \circ (D_{(t,s)}\varphi(X), D_{(t,s)}\varphi(Y)) \\ &= g(G^{-1}(t))^2 dt^2 (D_{(t,s)}\varphi(X), D_{(t,s)}\varphi(Y)) \\ &\quad + f(G^{-1}(t))^2 d\xi^2 (D_{(t,s)}\varphi(X), D_{(t,s)}\varphi(Y)). \end{aligned}$$

If X is a vector in the $d\xi$-direction, $D_{(t,s)}\varphi(X) = X$, since φ is radial. If X is in the dt-direction:

$$D_{(t,s)}\varphi(X) = \frac{d}{dx}|_{x=0} \varphi\big((t+x)X\big) = \underbrace{\frac{1}{g(G^{-1}(t))}}_{=\phi'(t)} \cdot X.$$

So, we can conclude our computation with

$$\begin{aligned}\tilde{h}(X,Y) &= \begin{cases} g(G^{-1}(t))^2 \cdot dt^2\Big(\frac{1}{g(G^{-1}(t))}\cdot X, \frac{1}{g(G^{-1}(t))}\cdot Y\Big) & \text{if } X,Y \text{ are in } dt\text{-direction}\\ f(G^{-1}(t))^2\cdot d\xi^2\big(X,Y\big) & \text{if } X,Y \text{ are in } d\xi\text{-direction}\\ 0 & \text{else}\end{cases}\\ &= \left(\frac{g(G^{-1}(t))}{g(G^{-1}(t))}\right)^2 dt^2 + f(G^{-1}(t))^2 d\xi^2 = dt^2 + f(G^{-1}(t))^2 d\xi^2. \qquad (5.2)\end{aligned}$$

$\tilde{h}$ is a metric on $D^q_{T(h)}$, where $T(h) = G(T_0)$, which depends continuously on g and therefore on h. Thus, we can assume without loss of generality that $g \equiv 1$ for the computations on scalar curvature.

Proposition 5.3 ([2, p. 269])**.** *If we identify $\{x \in \mathbb{R}^q : 0 < \|x\| < T\}$ with $(0,T)\times S^{q-1}$ in polar coordinates, the smooth Riemannian metric $dt^2 + f^2 d\xi^2$ extends to a smooth Riemannian metric on $\{x \in \mathbb{R}^q : \|x\| < T\}$ if and only if f is the restriction of a smooth odd function on $(-T,T)$ to $(0,T)$ with $f'(0) = 1$.*

The computation of the scalar curvature of a warped product metric can be found in [3, pp. 11-12] and [14, p. 13] and they are based upon [2, pp. 265-270]:

Proposition 5.4. *The scalar curvature of the metric $h = dt^2 + f^2 d\xi^2$ is given by*

$$\kappa = (q-1)\left((q-2)\frac{1-f'^2}{f^2} - 2\frac{f''}{f}\right). \qquad (5.5)$$

Corollary 5.6. *The scalar curvature of the metric $h = g^2dt^2 + f^2d\xi^2$ is given by*

$$\kappa = \frac{(q-1)}{f^2g^3}\Big((q-2)(g^3 - f'^2g) - 2f''fg + f'fg'\Big). \qquad (5.7)$$

Proof. The scalar curvature of the metric $h = g(t)^2dt^2 + f^2d\xi^2$ at a point t is the same as the scalar curvature of $h = dt^2 + \tilde{f}^2d\xi^2$ with $\tilde{f} = f \circ G^{-1}$ at the point $G^{-1}(t)$ (see 5.2). Then

$$\begin{aligned}\kappa &= (q-1)\left((q-2)\frac{1-(f\circ G^{-1})'^2}{(f\circ G^{-1})^2} - 2\frac{(f\circ G^{-1})''}{f\circ G^{-1}}\right)\\ &= \frac{(q-1)}{f^2g^3}\Big((q-2)(g^3 - f'^2g) - 2f''fg + f'fg'\Big).\end{aligned}$$

□

5.2 Deformation of warped metrics

5.2.1 Creating a collar

We fix a smooth function θ_0 on $\mathbb{R}$, such that θ_0 is 0 on $(-\infty, 0] \cup [\frac{1}{20}, \infty)$, $0 \le \theta_0 \le 1$ and $\theta_0 \ne 0$. We then define a smooth function C, depending on $(C_1, t^*) \in \mathbb{R}^2$:

$$C(C_1,t^*) = C_1\int_0^{\frac{t^*}{20}} \theta_0\Big(\frac{\sigma}{t^*}\Big)d\sigma = C_1 \cdot t^* \underbrace{\int_0^{\frac{1}{20}} \theta_0(\tilde{\sigma})d\tilde{\sigma}}_{=const.} = C_1 \cdot t^* \cdot const. \qquad (5.8)$$

Without loss of generality we may assume that θ_0 is small enough, i.e. we may assume that $C(C_1, t^*) < 1$ for all $C_1 \in [0,1]$ and $t^* \in [0, T_0/2]$.

Furthermore, we fix a family of strictly increasing Diffeomorphisms $\phi_{t_1,t_2,t_3} : \mathbb{R} \to \mathbb{R}$, continuously depending on $t_3 > t_2 > t_1$, such that

1. $\phi_{t_1,t_2,t_3}(0) = 0$
2. $\phi_{t_1,t_2,t_3}(t_2) = t_3$
3. $\phi'_{t_1,t_2,t_3} = 1$ on $[0, t_1] \cup [\frac{t_1+t_2}{2}, \infty)$.

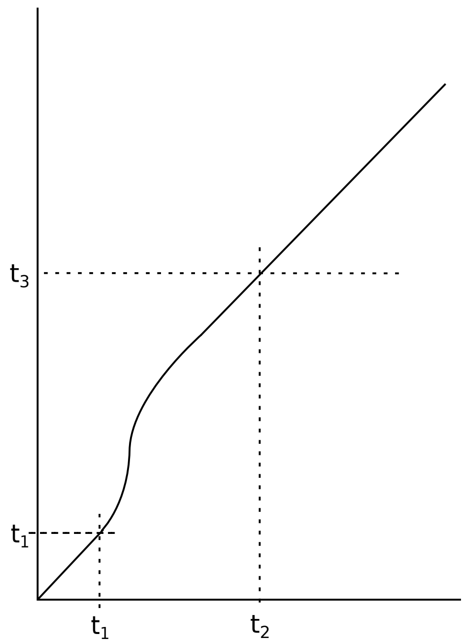


Figure 5.1: The rescaling diffeomorphism ϕ_{t_1,t_2,t_3}

One can construct this family for instance by taking a bump-function ϕ_0 which is 0 on $(-\infty, 0] \cup [1, \infty)$ and then solving the differential equation

$$\begin{aligned} \phi'_{t_1,t_2,t_3}(t) &= 1 + \frac{1}{\int_0^1 \phi_0(s)ds} \phi_0\Big(\frac{t-t_1}{\frac{t_1+t_2}{2} - t_1}\Big)(t_3 - t_2), \\ \phi_{t_1,t_2,t_3}(0) &= 0. \end{aligned}$$

The solution is given by

$$\phi_{t_1,t_2,t_3}(t) = t \cdot \Big(1 + \frac{\int_{t_1}^t \phi_0\Big(\frac{s-t_1}{\frac{t_1+t_2}{2} - t_1}\Big)ds}{\int_0^1 \phi_0(s)ds}(t_3 - t_2)\Big).$$

This allows us to define a family of radial diffeomorphisms $\Phi_{t_1,t_2,t_3} : \mathbb{R}^q \to \mathbb{R}^q$, continuously depending on t_1, t_2, t_3

$$\Phi_{t_1,t_2,t_3}(x) = \frac{\phi_{t_1,t_2,t_3}(\|x\|)}{\|x\|} x, \tag{5.9}$$

which satisfies:

1. $\Phi_{t_1,t_2,t_3}(D^q_{t_2}) = D^q_{t_3}$
2. $\Phi_{t_1,t_2,t_3} = id$ on $0 \leq \|x\| \leq t_1$ and
3. $\Phi_{t_1,t_2,t_3} = (1 + t_3 - t_2) \cdot id$ on $\frac{t_2+t_1}{2} \leq \|x\| \leq \infty$.

Having constructed these diffeomorphisms, we are now able to define the deformation of warped metrics on varying size discs, because we can then scale those discs back, using this family of diffeomorphisms. In the next two lemmata we will therefore define deformations on $\mathbb{R}$ instead of $[0, T_0]$.

One can interpret the following two lemmata (5.10 & 5.11) as first pushing $\tilde{\psi}_0 = id_{\mathbb{R}}$ down a little (5.10), so that we get an interval inside of $[0, T_0]$, where it has a slope that is constant $\tilde{C} < 1$. This also means that we created an interval, where we can estimate $(f \circ \tilde{\psi})' \leq \tilde{C}$ (provided $f' \leq 1$), which will be helpful when estimating the scalar curvature using the curvature formula (5.5). In the second lemma (5.11), we will deform $\psi_0 = id_{\mathbb{R}}$, such that there is a point t with $f^{(n)}(\psi_1(t)) = 0$ for all $n \geq 1$. This can be interpreted as creating a point, where the metric has the form $dt^2 + r_0^2 d\xi$ for some $r_0 > 0$. In other words, we created a point, where the metric is the product of a straight line metric with the standard round metric of the sphere of radius r_0, which can be seen as a collar.

In [3, p. 13], Chernysh describes these lemmata as first creating a large amount of curvature arbitrarily close to the center of the disc (5.10) and then creating a collar (5.11).

We will prove these two lemmata after we have seen why we need them.

Lemma 5.10 ([3, p. 13]). *Let* $A_1 := \{(C_1, t^*, \alpha) \in [0,1] \times [0, T_0/2] \times [0, T_0/2] : \alpha < t^*/2\}$.

There are continuous functions

$$\tilde{\psi}\colon A_1 \times I \times \mathbb{R} \to \mathbb{R}$$

and

$$T_1\colon I \to \mathbb{R},$$

such that:

1. $\tilde{\psi}_{\lambda,\alpha}(0) = 0, \tilde{\psi}_{\lambda,\alpha}(T_1(\lambda)) = t^*$
2. $\tilde{\psi}_{0,\alpha} = id$
3. $\tilde{\psi}''_{\lambda,\alpha} > 0$ *only if* $\tilde{\psi}_{\lambda,\alpha} \in [\frac{8}{10}t^*, \frac{9}{10}t^*]$
4. $\tilde{\psi}'_{\lambda,\alpha} \geq 1 - C(C_1, t^*) \geq 0$
5. $\tilde{\psi}''_{\lambda,\alpha} \leq C_1$
6. $\tilde{\psi}'_{\lambda,\alpha} = 1 - \lambda \cdot C(C_1, t^*) > 0$ *when* $\tilde{\psi}_{\lambda,\alpha} \in [\frac{\alpha}{10}, \frac{8}{10}t^*]$
7. $\tilde{\psi}'_{\lambda,\alpha} = 1$ *on* $[0, \frac{\alpha}{20}]$ *and when* $\tilde{\psi}_{\lambda,\alpha} \in [\frac{9}{10}t^*, \infty)$.

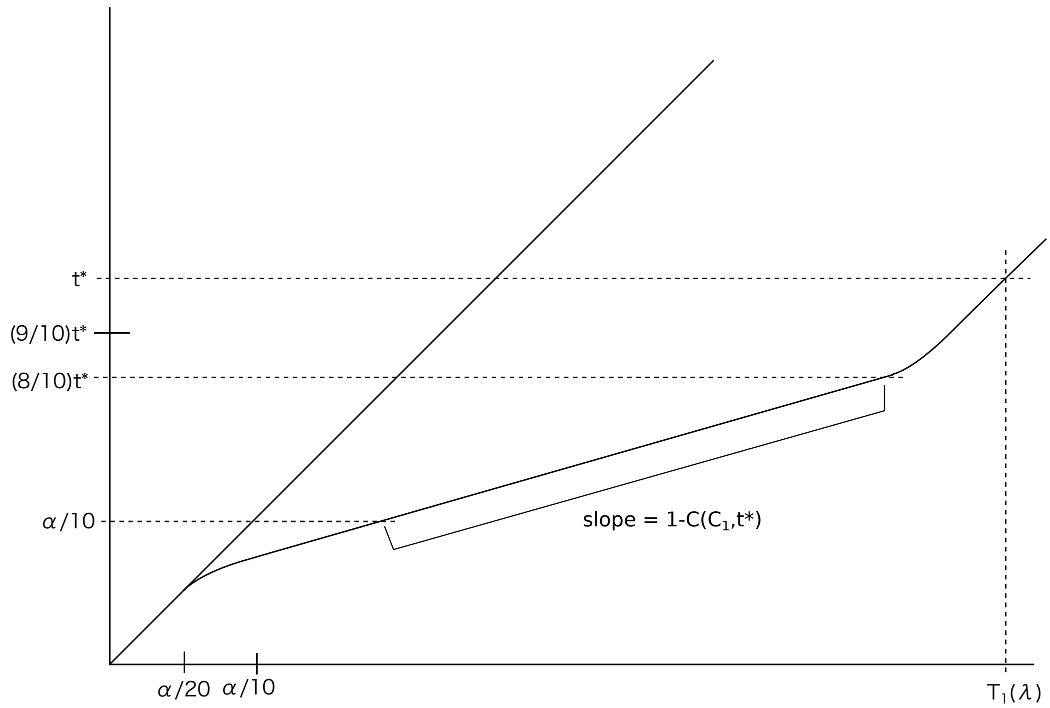


Figure 5.2: Creating a slope

Lemma 5.11 ([3, p. 15]). *There are continuous functions*

$$\begin{array}{rcl} \alpha\colon & (0,1]\times(0,T_0/2] & \to \mathbb{R} \\ & (C_{1,2},t^{**}) & \mapsto \alpha(C_{1,2},t^{**}) \\ T_2\colon & I\times[t^{**}/2,\infty) & \to \mathbb{R} \\ & (\lambda,T_1) & \mapsto T_2(\lambda,T_1) \\ \psi\colon & (0,1]\times(0,T_0/2]\times I\times\mathbb{R} & \to \mathbb{R} \\ & (C_{1,2},t^{**},\lambda,t) & \mapsto \psi_\lambda^{C_{1,2},t^{**}}(t)=\psi_\lambda(t), \end{array}$$

such that $\alpha(C_{1,2},t^{**})\in(0,t^{**}/2)$ *and:*

1. $\psi_\lambda(0)=0, \psi_\lambda(T_2(\lambda,T_1))=T_1$
2. $\psi_0=id$
3. $\psi_\lambda'(t)=1$ *for* $t\in[0,\frac{\alpha}{10}]$ *and if* $\psi_\lambda\in[\frac{9}{10}t^{**},\infty)$
4. $\psi_\lambda(t)''\leq\frac{C_{1,2}}{t}$
5. $0\leq\psi_\lambda'\leq 1$
6. $\psi_1^{(n)}\big(\alpha(C_{1,2},t^{**})\big)=0$ *for all* $n\geq 1$.

Note that the dependence on $(C_{1,2},t^{**})$ *is omitted in this notation of* ψ_λ.

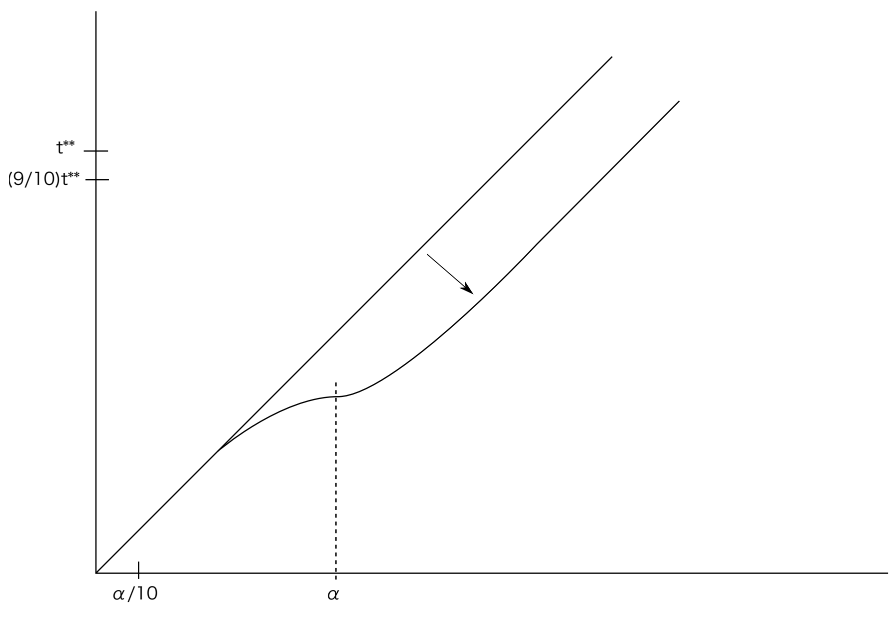


Figure 5.3: Creating a collar

Lemma 5.12 ([3, p. 17]). *Let $q \geq 3$. There is a continuous function*

$$\sigma\colon W_B \to (0, T_0/2]$$

and a continuous map

$$\Psi_1\colon W_B \times I \to W_B,$$

such that

1. $\Psi_1(\_, 0) = id_{W_B}$
2. $\Psi_1(h, \lambda) = h$ *near* $\partial D^q_{T_0}$
3. $\Psi_1(h, 1) = \tilde{h} = dt^2 + f^2 d\xi^2$ *satisfies* $f^{(n)}(\sigma(h)) = 0$ *for all* $n \geq 1$
4. $0 \leq f' \leq 1$ *and* $f'' \leq 0$ *on* $[0, \sigma(h)]$.

Proof. This proof is all about choosing constants and having functions satisfying certain properties. Therefore, it is very technical. As we said, we will prove the existence of certain functions afterwards (5.10 & 5.11), when it is clear what they are needed for.

We write a metric $h \in W_B$ as a pair $(1, f)$ with f being a function on $[0, T(h)]$. We then define the following continuous function:

$$\rho_1(t) := \min_{0 \leq \tau \leq t} \frac{f'(\tau)}{2f'(0)} T(h) = \min_{0 \leq \tau \leq t} \frac{f'(\tau)}{2} T(h).$$

This function is nonincreasing on $[0, T(h)/2]$ and satisfies $\rho_1(0) = T(h)/2$:

$$\rho_1(0) = \underbrace{\frac{f'(0)}{2f'(0)}}_{=1/2} T(h) = \frac{T(h)}{2}.$$

This means that there is a number $t_1 \in (0, T(h)/2]$, such that $t_1 = \rho_1(t_1)$ (intermediate value theorem). Furthermore, there exists a number $t_2 > 0$, such that $0 < f'(t) < 1$ and $f''(t) < 0$ for all $0 < t \leq t_2$, because if it didn't, the curvature formula (5.5) would imply that there is some point t with $\kappa < 0$, which is a contradiction. Let $t^* := \min(t_1, t_2, T_0/2)$. It follows that on $(0, t^*]$ we have:

$$\begin{aligned} 1 &> f'(t) > \min_{0 \leq \tau \leq t} f'(\tau) \geq \rho_1(t) \frac{2}{T(h)} > 0 \\ 0 &> f'' \end{aligned}$$

and

$$f''(0) = 0 = f(0), f'(0) = 1.$$

The last 2 equalities arise from (5.3) and from the fact that f and f'' are odd functions. Remember that B is the lower bound for the scalar curvature of the disc. The following constants will be chosen in a way that in the end we can estimate the scalar curvature of the metric $(1, f)$. They are not of any special interest on their own:

$$\begin{aligned}
\overline{B}' &:= \max_{[0,\frac{9}{10}t^*]} \frac{(q-1)(q-2)(1-f'^2) - Bf^2}{2(q-1)ff''} < 1 \\
B' &:= \max(\frac{1}{2}, \overline{B}') < 1 \\
B'' &:= \min_{[\frac{8}{10}t^*,\frac{9}{10}t^*]} \Big(\frac{q-2}{8}\frac{1-f'^2}{ff'} - \frac{f''}{4f'} - \frac{Bf}{8(q-1)f'}\Big) > 0.
\end{aligned}$$

We take the function $C(C_1, t^*)$ from (5.8) and choose $\tilde{C}_1$, such that $1 - C(\tilde{C}_1, t^*) = \sqrt{\frac{1+B'}{2}} > \sqrt{B'} \geq \sqrt{\overline{B}'}$.

$$\begin{aligned}
C_1 &= \min(\tilde{C}_1, B'') > 0 \\
t^{**} &= \min\left(\frac{8}{10}t^*, \sqrt{(q-1)(q-2)\frac{1-\big(1-C(C_1,t^*)\big)^2}{2B}}\right) > 0 \\
C_{1,2} &= \min\left(\frac{q-2}{4}\Big(1-\big(1-C(C_1,t^*)\big)^2\Big) - \frac{B}{4(q-1)}t^{**2}, 1\right) > 0.
\end{aligned}$$

Let's check whether the above inequalities hold. Our main tool here is the curvature formula (5.5):

$$\begin{aligned}
(q-1)(q-2)\frac{1-f'^2}{f^2} - 2(q-1)\frac{f''}{f} &> B \\
\iff (q-1)(q-2)\frac{1-f'^2}{f} - Bf &> 2(q-1)f'' \\
\underset{\text{on } (0,t^*]}{\overset{f''<0}{\iff}} \frac{(q-1)(q-2)(1-f'^2) - Bf^2}{2(q-1)f''f} &< 1 \\
\Rightarrow \overline{B}' &< 1,
\end{aligned}$$

$$\begin{aligned}
(q-1)(q-2)\frac{1-f'^2}{f^2} - 2(q-1)\frac{f''}{f} &> B \\
\underset{\text{on } (0,t^*]}{\overset{f>0}{\iff}} \frac{q-2}{8}\frac{1-f'^2}{f} - \frac{f''}{4} &> \frac{Bf}{8(q-1)} \\
\iff \frac{q-2}{8}\frac{1-f'^2}{f} - \frac{f''}{4} - \frac{Bf}{8(q-1)} &> 0 \\
\overset{f'>0}{\Rightarrow} B'' &> 0.
\end{aligned}$$

And finally,

$$\begin{aligned}
& \frac{q-2}{4}\Big(1-\big(1-C(C_1,t^*)\big)^2\Big)-\frac{B}{4(q-1)}t^{**2} \\
\geq\ & \frac{q-2}{4}\Big(1-\big(1-C(C_1,t^*)\big)^2\Big)-\frac{B}{4(q-1)}(q-1)(q-2)\frac{1-\big(1-C(C_1,t^*)\big)^2}{2B} \\
=\ & \frac{q-2}{4}\Big(1-\big(1-C(C_1,t^*)\big)^2\Big)-\frac{q-2}{8}\Big(1-\big(1-C(C_1,t^*)\big)^2\Big) \\
=\ & \frac{q-2}{8}\Big(1-\big(1-C(C_1,t^*)\big)^2\Big)>0 \\
& \Rightarrow C_{1,2}>0.
\end{aligned}$$

We take the function $\alpha = \alpha(C_{1,2}, t^{**})$ from (5.10). Note that all these constants (including the functions C, α) depend continuously on the metric h, which can be easily verified by taking a converging sequence of metrics and checking the constants for convergence.

We take the functions $\tilde{\psi}_{\lambda,\alpha}\colon \mathbb{R}\to\mathbb{R}$ and $\psi_\lambda\colon\mathbb{R}\to\mathbb{R}$ from (5.10 & 5.11). If we write a warped metric h as $h=(1,f)$, the desired deformation is given by:

$$\tilde{\Psi}_1(h,\lambda)=\begin{cases}\Big(1,f\big(\tilde{\psi}_{2\lambda,\alpha}(t)\big)\Big) & \lambda\in[0,\frac{1}{2}]\\ \Big(1,f\big(\tilde{\psi}_{1,\alpha}(\psi_{2\lambda-1}(t))\big)\Big) & \lambda\in[\frac{1}{2},1].\end{cases} \tag{5.13}$$

This is a continuous family of metrics on $\mathbb{R}^q$. Using our family of diffeomorphisms from (5.9), we can rescale $\mathbb{R}^q$, so that we get a family of metrics on $D^q_{T_0}$:

$$\Psi_1(h,\lambda)=\begin{cases}\underbrace{\Phi^*_{\frac{9}{10}t^*,t^*,T_1(2\lambda)}\underbrace{\Big(1,f\big(\tilde{\psi}_{2\lambda,\alpha}(t)\big)\Big)}_{\text{metric on } D^q_{\tilde{T}_1(\lambda)}}}_{\text{metric on } D^q_{T_0}} & \lambda\in[0,\frac{1}{2}]\\ \underbrace{\Phi^*_{\frac{9}{10}t^*,t^*,T_1(1)}\Phi^*_{\frac{9}{10}t^*,T_1(1),T_2((2\lambda-1),T_1(1))}\underbrace{\Big(1,f\big(\tilde{\psi}_{1,\alpha}(\psi_{2\lambda-1}(t))\big)\Big)}_{\text{metric on } D^q_{\tilde{T}_2(2\lambda-1)}}}_{\text{metric on } D^q_{T_0}}, & \lambda\in[\frac{1}{2},1]\end{cases} \tag{5.14}$$

where

$$\tilde{T}_1(\lambda)=\phi_{\frac{9}{10}t^*,t^*,T_1(\lambda)}(T_0)$$

and

$$\tilde{T}_2(\lambda)=\tilde{T}_1(\lambda)=\phi_{\frac{9}{10}t^*,T_1(1),T_2(\lambda,T_1(1))}\circ\phi_{\frac{9}{10}t^*,t^*,T_1(1)}(T_0).$$

From (5.10, 1 & 5.11, 1) we see that this really is a deformation of $\mathcal{R}(D^q_{T_0})$ into itself and by (5.10 & 5.11) this is continuous. Furthermore, it is constant near the boundary, since the derivative of our family of diffeomorphisms is the identity there and the functions $\psi, \tilde{\psi}$ end in straight lines with slope 1. On $[0, \frac{9}{10}t^*]$ the diffeomorphisms

$\phi_{(\dots)}$ are all equal to $id_{[0,\frac{9}{10}t^*]}$ and thus the metrics have the form $dt^2 + f(\tilde{\psi})^2 d\xi^2$ or $dt^2 + f(\tilde{\psi} \circ \psi)^2 d\xi^2$ here.

We will abbreviate $\tilde{\psi}_{\lambda,\alpha}$ and ψ_λ by $\tilde{\psi}$ and ψ. In this notation the dependence on $C_1, t^*, C_{1,2}, t^{**}, \lambda, \alpha$ will be implied.

What remains to be done now is to verify that the scalar curvature is greater than B during the deformation. These computations are performed for the metrics on $D^q_{\tilde{T}_i(\lambda)}$. The scalar curvature of $(1, f(\tilde{\psi}))$ (see (5.5)) is given by:

$$\begin{aligned}\kappa &= (q-1)\left((q-2)\frac{1-\left(\frac{d}{dt}f(\tilde{\psi})\right)^2}{f(\tilde{\psi})^2} - 2\frac{\frac{d^2}{dt^2}f(\tilde{\psi})}{f(\tilde{\psi})}\right) \\ &= (q-1)\left((q-2)\frac{1-f'(\tilde{\psi})^2\tilde{\psi}'^2}{f(\tilde{\psi})^2} - 2\frac{\frac{d}{dt}\left(f'(\tilde{\psi})\tilde{\psi}'\right)}{f(\tilde{\psi})}\right) \\ &= (q-1)\left((q-2)\frac{1-f'(\tilde{\psi})^2\tilde{\psi}'^2}{f(\tilde{\psi})^2} - 2\frac{f''(\tilde{\psi})\tilde{\psi}'^2}{f(\tilde{\psi})} - 2\frac{f'(\tilde{\psi})\tilde{\psi}''}{f(\tilde{\psi})}\right). \qquad (5.15)\end{aligned}$$

Until $\tilde{\psi}$ reaches $\frac{8}{10}t^*$, $\tilde{\psi}'' \leq 0$ holds (5.10, 3) and therefore $-2\frac{f'(\tilde{\psi})\tilde{\psi}''}{f(\tilde{\psi})} \geq 0$.

$$\begin{aligned}\kappa \quad &= \quad (q-1)\left((q-2)\frac{1-f'(\tilde{\psi})^2\tilde{\psi}'^2}{f^2(\tilde{\psi})} - 2\frac{f''(\tilde{\psi})\tilde{\psi}'^2}{f(\tilde{\psi})} - 2\frac{f'(\tilde{\psi})\tilde{\psi}''}{f(\tilde{\psi})}\right) \overset{!}{>} B \\ &\Leftarrow \quad (q-1)\left((q-2)\frac{1-f'(\tilde{\psi})^2\tilde{\psi}'^2}{f(\tilde{\psi})^2} - 2\frac{f''(\tilde{\psi})\tilde{\psi}'^2}{f(\tilde{\psi})}\right) > B \\ &\iff \quad (q-1)(q-2)\frac{1-f'(\tilde{\psi})^2\tilde{\psi}'^2}{f(\tilde{\psi})^2} - B > 2(q-1)\frac{f''(\tilde{\psi})\tilde{\psi}'^2}{f(\tilde{\psi})} \\ &\overset{\text{wlog, } f''<0}{\iff} \quad \frac{(q-1)(q-2)(1-f'(\tilde{\psi})^2\tilde{\psi}'^2) - Bf(\tilde{\psi})^2}{2(q-1)f''(\tilde{\psi})f(\tilde{\psi})} < \tilde{\psi}'^2.\end{aligned}$$

Since $\tilde{\psi}'^2 \geq \left(1 - C(C_1, t^*)\right)^2 > \overline{B}'$ (5.10, 4), the above equation is true.

If $\tilde{\psi} \in [\frac{8}{10}t^*, \frac{9}{10}t^*]$, $\tilde{\psi}''$ might be positive. On this interval, we have to verify:

$$\begin{aligned}\kappa \quad &= \quad (q-1)\left((q-2)\frac{1-f'(\tilde{\psi})^2\tilde{\psi}'^2}{f(\tilde{\psi})^2} - 2\frac{f''(\tilde{\psi})\tilde{\psi}'^2}{f(\tilde{\psi})} - 2\frac{f'(\tilde{\psi})\tilde{\psi}''}{f(\tilde{\psi})}\right) \overset{!}{>} B \\ &\iff \quad (q-1)\left((q-2)\frac{1-f'(\tilde{\psi})^2\tilde{\psi}'^2}{f(\tilde{\psi})^2} - 2\frac{f''(\tilde{\psi})\tilde{\psi}'^2}{f(\tilde{\psi})}\right) - B > 2(q-1)\frac{f'(\tilde{\psi})\tilde{\psi}''}{f(\tilde{\psi})} \\ &\iff \quad \frac{q-2}{2}\frac{1-f'(\tilde{\psi})^2\tilde{\psi}'^2}{f(\tilde{\psi})f'(\tilde{\psi})} - \frac{f''(\tilde{\psi})\tilde{\psi}'^2}{f'(\tilde{\psi})} - \frac{Bf(\tilde{\psi})}{2(q-1)f'(\tilde{\psi})} > \tilde{\psi}''.\end{aligned}$$

Again, from (5.10, 4) we can conclude:

$$
\begin{aligned}
& \frac{q-2}{2}\frac{1-f'(\tilde\psi)^2\tilde\psi'^2}{f(\tilde\psi)f'(\tilde\psi)} - \frac{f''(\tilde\psi)\tilde\psi'^2}{f'(\tilde\psi)} - \frac{Bf(\tilde\psi)}{2(q-1)f'(\tilde\psi)} \\
\underset{f''<0,\ \tilde\psi'^2\geq\frac{1+B'}{2}}{\overset{\tilde\psi'\leq 1}{\geq}} \quad & \frac{q-2}{2}\frac{1-f'(\tilde\psi)^2}{f(\tilde\psi)f'(\tilde\psi)} - \frac{f''(\tilde\psi)}{f'(\tilde\psi)}\left(\frac{1+B'}{2}\right) - \frac{Bf(\tilde\psi)}{2(q-1)f'(\tilde\psi)} \\
\underset{B'\geq\overline{B}'}{\geq} \quad & \frac{q-2}{2}\frac{1-f'(\tilde\psi)^2}{f(\tilde\psi)f'(\tilde\psi)} - \frac{f''(\tilde\psi)}{2f'(\tilde\psi)} \\
& -\frac{(q-1)(q-2)(1-f'(\tilde\psi)^2)-Bf(\tilde\psi)^2}{4(q-1)f(\tilde\psi)f'(\tilde\psi)} - \frac{Bf(\tilde\psi)}{2(q-1)f'(\tilde\psi)} \\
= \quad & \frac{q-2}{4}\frac{1-f'(\tilde\psi)^2}{f(\tilde\psi)f'(\tilde\psi)} - \frac{f''(\tilde\psi)}{2f'(\tilde\psi)} - \frac{Bf(\tilde\psi)}{4(q-1)f'(\tilde\psi)} \\
> \quad & B'' \geq C_1 \overset{(5.10,\ 5)}{\geq} \tilde\psi''
\end{aligned}
$$

Since $\tilde\psi' = 1$ when $\tilde\psi \in [\frac{9}{10}t^*, \infty)$ we get that

$$
\begin{aligned}
\kappa &= (q-1)\left((q-2)\frac{1-f'(\tilde\psi)^2\ \overbrace{\tilde\psi'^2}^{=1}}{f^2(\tilde\psi)} - 2\frac{f''(\tilde\psi)\ \overbrace{\tilde\psi'^2}^{=1}}{f(\tilde\psi)} - \underbrace{2\frac{f'(\tilde\psi)\tilde\psi''}{f(\tilde\psi)}}_{=0}\right) \\
&= (q-1)\left((q-2)\frac{1-f'(\tilde\psi)^2}{f^2(\tilde\psi)} - 2\frac{f''(\tilde\psi)}{f(\tilde\psi)}\right) > B.
\end{aligned}
$$

which is clear, as $h \in W_B$. This concludes the curvature computation during the first part of the deformation.

For the computations on the second part of the deformation we will write $\tilde f = f(\tilde\psi)$ and $\psi = \psi_\lambda$. By (5.11, 3) we only need to estimate the scalar curvature when $\psi \in [\frac{\alpha}{10}, t^{**}]$. First we note that the inequality $0 < \tilde f' \leq \tilde C = 1 - C(C_1, t^*) < 1$ holds here, because $\tilde\psi'_{1,\alpha} = 1 - C(C_1, t^*)$ and $f' \in (0,1)$ on this interval (see (5.10, 6)). It again suffices to show

$$
\frac{q-2}{2}\frac{1-\tilde f'(\psi)^2\psi'^2}{\tilde f(\psi)\tilde f'(\psi)} - \frac{\tilde f''(\psi)\psi'^2}{\tilde f'(\psi)} - \frac{B\tilde f(\psi)}{2(q-1)\tilde f'(\psi)} > \psi''.
$$

So,

$$\begin{array}{cl}
& \frac{q-2}{2}\frac{1-\tilde f'(\psi)^2\psi'^2}{\tilde f(\psi)\tilde f'(\psi)} - \frac{\tilde f''(\psi)\psi'^2}{\tilde f'(\psi)} - \frac{B\tilde f(\psi)}{2(q-1)\tilde f'(\psi)} \\
\overset{\tilde f'(\psi)\psi'\le \tilde C}{\underset{\tilde f''(\psi)\psi'^2\le 0}{\ge}} & \frac{1}{\tilde f'(\psi)}\Big(\frac{q-2}{2}\frac{1-\tilde C^2}{\tilde f(\psi)} - \frac{B\tilde f(\psi)}{2(q-1)}\Big) \\
\overset{\tilde f'(\psi)\le 1}{\ge} & \frac{(q-2)(1-\tilde C^2) - \frac{B}{(q-1)}\tilde f(\psi)^2}{2\tilde f(\psi)} \\
\overset{\tilde f(\psi(t))\le t}{\underset{\tilde f(\psi(t))\le t^{**}}{\ge}} & \frac{(q-2)(1-\tilde C^2) - \frac{B}{(q-1)}t^{**2}}{2t} \\
= & \frac{2}{t}\left(\frac{q-2}{4}\Big(1-\big(1-C(C_1,t^*)\big)^2\Big) - \frac{B}{4(q-1)}t^{**2}\right) \\
\ge & 2\frac{C_{1,2}}{t} \overset{(5.11,\ 4)}{>} \psi''.
\end{array}$$

We now define $\sigma(h) = \alpha \in [0, t^*/2]$ (Recall that on this interval, $\Phi_{\frac{9}{10}t^*,t^*,T_2(\lambda)}$ is the identity) and get that $\Psi_1(h,1) = (1,\tilde f)$ satisfies $0 \le \tilde f' \le 1$, $\tilde f'' \le 0$ on $[0,\sigma]$ and

$$\tilde f^{(n)}(\sigma(h)) = 0 \text{ for all } n \ge 1,$$

which completes this proof. □

Proof of lemma 5.10. We take the function θ_0 from (5.8). The strategy here is to construct a function $\theta_{\lambda,\alpha}(t)$ and then to take $\tilde\psi_{\lambda,\alpha}$ to be the solution of the second order differential equation

$$\begin{aligned}
\tilde\psi''_{\lambda,\alpha}(t) &= \theta_{\lambda,\alpha}(t)\\
\tilde\psi_{\lambda,\alpha}(0) &= 0\\
\tilde\psi'_{\lambda,\alpha}(0) &= 1.
\end{aligned}$$

Since we want to start with $\tilde\psi'_{\lambda,\alpha} \equiv 1$ (5.6, 7) the second derivative has to start with an interval on which it is 0. When $\tilde\psi_{\lambda,\alpha} \in [\frac{\alpha}{10}, \frac{8}{10}t^*]$ we want $\tilde\psi_{\lambda,\alpha}$ to satisfy $\tilde\psi'_{\lambda,\alpha} = 1-\lambda C(C_1,t^*)$, so we have to perform a downwards bend: $\tilde\psi''_{\lambda,\alpha} = -\frac{1}{\alpha}\lambda C_1\theta_0(\frac{t}{\alpha} - \frac{1}{20})t^*$ on $[\frac{\alpha}{20}, \frac{\alpha}{10}]$. Our next requirement is $\tilde\psi'_{\lambda,\alpha} = 1$, while $\tilde\psi_{\lambda,\alpha} \in [\frac{9}{10}t^*, \infty)$. Since the slope we have is greater than 0, we deduce that there is a point t_λ, depending continuously on all our parameters, such that $\tilde\psi_{\lambda,\alpha}(t_\lambda) = \frac{8}{10}t^*$. From that point on, we can perform an upwards bend $\lambda C_1\theta_0(\frac{t-t_\lambda}{t^*})$ on $[t_\lambda, t_\lambda + \frac{t^*}{20}]$. Putting all these pieces together, we get

$$\theta_{\lambda,\alpha}(t) = \begin{cases} -\frac{1}{\alpha}\lambda C_1\theta_0\big(\frac{t}{\alpha} - \frac{1}{20}\big)t^* & \text{on } [\frac{\alpha}{20}, \frac{\alpha}{10}] \\ \lambda C_1\theta_0\big(\frac{t-t_\lambda}{t^*}\big) & \text{on } [t_\lambda, t_\lambda + \frac{t^*}{20}] \\ 0 & \text{otherwise} \end{cases}$$

and we take $\tilde{\psi}_{\lambda,\alpha}$ to be the solution of our second order differential equation

$$\begin{aligned}\tilde{\psi}''_{\lambda,\alpha}(t) &= \theta_{\lambda,\alpha}(t)\\ \tilde{\psi}_{\lambda,\alpha}(0) &= 0\\ \tilde{\psi}'_{\lambda,\alpha}(0) &= 1\end{aligned}$$

and $T_1(\lambda) = \tilde{\psi}^{-1}_{\lambda,\alpha}(t^*)$. By (2.57) this is continuous in all our variables. The explicit solution is

$$\psi_\lambda(t) = t + \int_0^t \int_0^s \theta_{\lambda,\alpha}(r) dr ds.$$

All that is left to do is to verify the 7 conditions above:

1., 2., & 5. are obvious from our construction.

3. Here it suffices to show $\tilde{\psi}_{\lambda,\alpha}(t_\lambda + \frac{t^*}{20}) \leq \frac{9}{10} t^*$:

$$\tilde{\psi}_{\lambda,\alpha}(t_\lambda + \frac{t^*}{20}) = t_\lambda + \frac{t^*}{20} + \int_0^{t_\lambda + \frac{t^*}{20}} \int_0^s \theta_{\lambda,\alpha}(r) dr ds$$

$$= \underbrace{t_\lambda + \int_0^{t_\lambda} \int_0^s \theta_{\lambda,\alpha}(r) dr ds}_{=\tilde{\psi}_{\lambda,\alpha}(t_\lambda) = \frac{8}{10} t^*} + \underbrace{\frac{t^*}{20} + \int_{t_\lambda}^{t_\lambda + \frac{t^*}{20}} \underbrace{\int_0^s \theta_{\lambda,\alpha}(r) dr}_{\leq \lambda \cdot C(C_1, t^*) \leq 1} ds}_{\leq \frac{9}{10} t^*} \leq \frac{9}{10} t^*$$

4. $$\tilde{\psi}'_{\lambda,\alpha}(t) = 1 + \int_0^t -\frac{1}{\alpha} \lambda C_1 \theta_0 \Big(\frac{s}{\alpha} - \frac{1}{20}\Big) t^* ds + \int_0^t \lambda C_1 \theta_0 \Big(\frac{s - t_\lambda}{t^*}\Big) ds$$

$$\geq 1 + \int_{\frac{\alpha}{20}}^{\frac{\alpha}{10}} -\frac{1}{\alpha} \lambda C_1 \theta_0 \Big(\frac{s}{\alpha} - \frac{1}{20}\Big) t^* ds = 1 - \lambda C(C_1, t^*)$$

6. follows from the same computation.

7. follows from

$$\int_{\frac{\alpha}{20}}^{\frac{\alpha}{10}} \frac{2}{\alpha} \lambda C_1 \theta_0 \Big(\frac{s}{\alpha} - \frac{1}{20}\Big) t^* ds = \int_{t_\lambda}^{t_\lambda + \frac{t^*}{20}} \lambda C_1 \theta_0 \Big(\frac{s - t_\lambda}{t^*}\Big) ds.$$

□

For the proof of (5.11) we need a little preparation.

Lemma 5.16 ([3, p. 15]). *Let $a < b \in \mathbb{R}, c \in (0,1)$ and $f \geq 0, f \neq 0$ a continuous function on $[a,b]$. Then there is a smooth cutoff function ϕ on $[a,b]$, i.e. smooth, increasing functions on $\mathbb{R}$, such that $\phi(t) = 0$ for $t \leq a$ and $\phi(t) = 1$ for $t \geq b$, such that*

$$\int_a^b \phi(t) f(t) dt = c \int_a^b f(t) dt.$$

ϕ depends continuously on a, b and c. The same holds for inverse cutoff functions, i.e. for smooth, decreasing functions $1 \geq \phi \geq 0$, such that $\phi(t) = 1$ for $t \leq a$ and $\phi(t) = 0$ for $t \geq b$.

Proof. Let $\tilde{\phi}$ be a fixed cutoff function on $[0,1]$. Then $\tilde{\phi}\big(\frac{t-a}{b-a}\big)$ is a cutoff function on $[a,b]$ and so is $\phi_\mu = \tilde{\phi}\big(\big(\frac{t-a}{b-a}\big)^\mu\big)$ for any $\mu \in (0,\infty)$. We will show that we can choose μ, so that the equation above is fulfilled and that our choice of μ is continuous in a, b and c.

Consider the smooth, strictly decreasing function

$$\begin{aligned} H\colon (0,\infty) &\to \mathbb{R} \\ \mu &\mapsto \int_a^b \Big(\phi\Big(\Big(\frac{t-a}{b-a}\Big)^\mu\Big) - c\Big) f(t)dt. \end{aligned}$$

Then $H(\mu) < 0$ for μ big enough and $H(\mu) > 0$ for μ close enough to 0. By the intermediate value theorem, there is a unique (remember that H is increasing) $\mu_0 \in (0,\infty)$, such that $H(\mu_0)$=0, which means we have found our cutoff function. Now let (a_n, b_n, c_n) be a sequence converging to (a,b,c) and let μ_n, μ be the numbers constructed above belonging to (a_n, b_n, c_n) and (a,b,c). Then the function

$$\begin{aligned} G\colon \mathbb{R}^3 &\to \mathbb{R} \\ (a,b,c) &\mapsto \int_a^b \Big(\phi\Big(\Big(\frac{t-a}{b-a}\Big)^\mu\Big) - c\Big) f(t)dt. \end{aligned}$$

is continuous, which means $G(a_n, b_n, c_n) \to G(a,b,c) = 0$. Let $\nu \in [0,\infty]$ be an accumulation point of this sequence and let μ_{n_k} be a subsequence converging to ν.

$$\begin{aligned} H(\mu_n) - G(a_n,b_n,c_n) &= H(\mu_n) - \underbrace{H(\mu) + G(a,b,c)}_{=0} - G(a_n,b_n,c_n) \\ &= \underbrace{H(\mu_n) - H(\mu)}_{\to 0} + \underbrace{G(a,b,c) - G(a_n,b_n,c_n)}_{\to 0} \to 0 \\ \Rightarrow \lim_{n\to\infty} H(\mu_n) &= \lim_{n\to\infty} G(a_n,b_n,c_n). \end{aligned}$$

So

$$H(\nu) = \lim_{k\to\infty} H(\mu_{n_k}) = \lim_{k\to\infty} G(a_{n_k}, b_{n_k}, c_{n_k}) = 0$$

and we get that $H(\nu) = 0 = H(\mu)$. But since H is strictly increasing we deduce $\mu = \nu$ and therefore $\mu_n \to \mu$.

The proof for inverse cutoff functions works in the same way. $\square$

Proof of Lemma 5.11. We again will construct ψ_λ as the solution of a differential equation.

$$\psi''_\lambda(t) = \theta_\lambda(t)$$

with the initial conditions $\psi'_\lambda(0) = 1$ and $\psi_\lambda(0) = 0$. What we have to do now, is construct the function θ_λ.

We set

$$\begin{aligned} \alpha &:= \frac{9}{20} t^{**} e^{-\frac{1.1}{C_{1,2}}} \\ t_0 &:= \alpha \exp(1.1/C_{1,2}). \end{aligned}$$

Then we have

$$\begin{aligned}
\alpha &= \underbrace{\frac{9}{20}}_{<1/2} t^{**} \underbrace{e^{-\frac{1.1}{C_{1,2}}}}_{<1} < \frac{1}{2} t^{**} \\
t_0 &= \alpha \exp(1.1/C_{1,2}) = \frac{9}{20} t^{**} \underbrace{e^{-\frac{1.1}{C_{1,2}}} \exp(1.1/C_{1,2})}_{=1} = \frac{9}{20} t^{**} < \frac{9}{10} t^{**}
\end{aligned}$$

and

$$\int_{\alpha}^{t_0} \frac{C_{1,2}}{t} dt = C_{1,2} \ln\Big(\frac{t_0}{\alpha}\Big) = C_{1,2} \ln\Big(\exp\big(1.1/C_{1,2}\big)\Big) = 1.1$$

Let $\alpha_1 < t_1 \in [\alpha, t_0]$ be the unique numbers, such that

$$\int_{\alpha}^{\alpha_1} \frac{C_{1,2}}{t} dt = 0.1 = \int_{t_0}^{t_1} \frac{C_{1,2}}{t} dt.$$

Then $\int_{\alpha_1}^{t_1} \frac{C_{1,2}}{t} dt = 0.9$ and by (5.16) there are cutoff functions ϕ_1, ϕ_2 depending continuously on $C_{1,2}$ and t^{**}, such that

$$\int_{\alpha}^{\alpha_1} \phi_1(t) \frac{C_{1,2}}{t} dt = 0.05 = \int_{t_0}^{t_1} \phi_2(t) \frac{C_{1,2}}{t} dt.$$

Finally, we are able to define

$$\tilde{\theta}_1(t) = \begin{cases} \phi_1(t) \frac{C_{1,2}}{t} & \text{if } t \in [\alpha, \alpha_1] \\ \frac{C_{1,2}}{t} & \text{if } t \in [\alpha_1, t_1] \\ \phi_2(t) \frac{C_{1,2}}{t} & \text{if } t \in [t_1, t_0] \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\int \tilde{\theta}_1(t) dt = 1$. Let $0 \le \tilde{\theta}_2(t) \le 1$ be a smooth, nonnegative bump function on $[0, 1]$, such that $\int \tilde{\theta}_2(t) dt = 1$.

$$\theta_1(t) = \begin{cases} -\frac{10}{9\alpha} \tilde{\theta}_2\big(\frac{10t-\alpha}{9\alpha}\big) & \text{if } t \in [\frac{\alpha}{10}, \alpha] \\ \tilde{\theta}_1(t) & \text{if } t \in [\alpha, t_0] \\ 0 & \text{otherwise.} \end{cases}$$

This is continuous in $\alpha, C_{1,2}$ and t^{**} and of course smooth in t. θ_λ is then given by:

$$\theta_\lambda = \lambda \cdot \theta_1$$

and thus

$$\psi_\lambda(t) = t + \int_0^t \int_0^s \theta_\lambda(r) dr ds.$$

From (2.57) we know that this is continuous in $\lambda, \alpha, C_{1,2}$ and t^{**}. When ψ_λ reaches $T_1 \ge t^{**}/2 > \alpha$ we see that $\psi_\lambda' > 0$. Therefore it is injective here and it makes sense to define $T_2(\lambda, T_1) = \psi_\lambda^{-1}(T_1)$.

What's left to do, is to check the 6 requirements for this family of functions. 1, 2, 4 and 5 are obvious from our construction. The only things that remain to be shown are requirements 3 and 6.

Let's start with 3. $\psi'_\lambda(t) = 1$ on $[0, \alpha/10]$ is clear. To show $\psi'_\lambda(t) = 1$ when $\psi_\lambda \in [\frac{9}{10}t^*, \infty)$ it is sufficient to show that $\psi_\lambda(t_0) \leq \frac{9}{10}t^{**}$. Then for $\psi_\lambda(t) \geq \frac{9}{10}t^{**} \geq \psi_\lambda(t_0)$ we have $t \geq t_0$ and therefore

$$\begin{aligned}
\psi'_\lambda(t) &= 1 + \int_0^t \theta_\lambda(s)ds \\
&= 1 + \lambda\Big(\underbrace{\int_{\alpha/10}^{\alpha} -\frac{10}{9\alpha}\tilde{\theta}_2(\frac{10s-\alpha}{9\alpha})ds}_{=-1} + \underbrace{\int_\alpha^{t_0} \tilde{\theta}_1(s)ds}_{=1}\Big) = 1.
\end{aligned}$$

Note that

$$\int_\alpha^s \theta_\lambda(r)dr = -\int_s^\alpha \theta_\lambda(r)dr \leq \lambda\int_0^\alpha \frac{10}{9\alpha}\tilde{\theta}_2(\frac{10r-\alpha}{9\alpha})dr = \lambda \qquad \text{for } s \leq \alpha$$

$$\int_\alpha^s \theta_\lambda(r)dr = \lambda\int_\alpha^s \tilde{\theta}_1(r)dr \leq \lambda\int_\alpha^{t_0} \tilde{\theta}_1(s)ds = \lambda \qquad \text{for } s \geq \alpha$$

From our construction we know:

$$\begin{aligned}
\psi_\lambda(t_0) &= t_0 + \int_0^{t_0}\int_0^s \theta_\lambda(r)drds. \\
&= t_0 + \int_0^{t_0}\Big(\underbrace{\int_0^\alpha \theta_\lambda(r)dr}_{=-\lambda} + \int_\alpha^s \theta_\lambda(r)dr\Big)ds \\
&= t_0 - \lambda t_0 + \int_0^{t_0}\int_\alpha^s \theta_\lambda(r)drds \\
&= t_0 - \lambda t_0 + \underbrace{\int_0^\alpha \underbrace{\int_\alpha^s \theta_\lambda(r)dr}_{\leq\lambda}\, ds}_{\leq\lambda\alpha} + \int_\alpha^{t_0}\int_\alpha^s \theta_\lambda(r)drds \\
&\leq t_0 - \lambda(t_0-\alpha) + \underbrace{\int_\alpha^{t_0} \underbrace{\int_\alpha^s \theta_\lambda(r)dr}_{\leq 1}\, ds}_{\leq\lambda(t_0-\alpha)} \\
&\leq t_0 - \lambda(t_0-\alpha) + \lambda(t_0-\alpha) = t_0 \overset{\text{see above}}{<} \frac{9}{10}t^{**}.
\end{aligned}$$

Last but not least we consider 6:

$$\begin{aligned}
\psi'_1(\alpha) &= 1 + \int_0^\alpha \theta_1(s)ds = 1 + \underbrace{\int_0^\alpha -\frac{10}{9\alpha}\tilde{\theta}_2(\frac{10s-\alpha}{9\alpha})ds}_{=-1} = 0, \\
\psi_1^{(n)}(\alpha) &= \tilde{\theta}_1^{(n)}(\alpha) = \big(\phi_1(\alpha)\frac{C_{1,2}}{\alpha}\big)^{(n-2)} \\
&= \sum_{l=0}^{n-2}\binom{n-2}{l}\underbrace{\phi_1^{(l)}(\alpha)}_{=0}\Big(\frac{C_{1,2}}{\alpha}\Big)^{(n-2-l)} = 0,
\end{aligned}$$

because ϕ_1 is a smooth cutoff function. Thus, the function ψ_1 and every derivative vanish at α. $\square$

5.2.2 Deforming W into W^{loc}

Remark 5.17. *Our torpedo metric g_0 of radius T_0 is a warped metric in the disc. Let's assume that $g_0 \in W_B$, i.e. T_0 is so small that the scalar curvature of g_0 is greater than B. From now on we will write*

$$g_0 = dt^2 + \tilde{f}_0(t)^2 d\xi^2.$$

with warping function

$$\tilde{f}_0(t) = T_0 \cdot \sin\underbrace{\Big(\frac{\pi}{2}\frac{t}{T_0}\Big)}_{=\theta}.$$

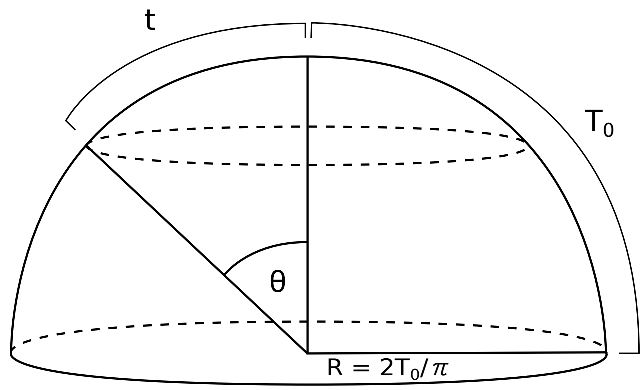


Figure 5.4: $D^q_{T_0}$ interpreted as the upper hemisphere of $S^q(T_0)$

Note that in this case $0 \leq \tilde{f}_0' \leq 1$, $\tilde{f}_0'' \leq 0$ and $\tilde{f}_0'' < 0$ on $(0, T_0]$.

Definition 5.18. Let $h \in W_B$. We call h a local torpedo metric if, for some number $c \in [0, T_0]$, the metric h is a warped metric of the form

$$h = \Big(\frac{T_0}{c}\Big)^2 dt^2 + \tilde{f}_0\Big(\frac{T_0}{c}t\Big)^2 d\xi^2$$

in the disc D^q_c. In other words, the metric h is the pullback of our fixed torpedo metric g_0 under the linear map $\phi = (T_0/c) \cdot id_{\mathbb{R}^q}$ inside D^q_c, i.e. it is a torpedo metric on D^q_c of radius T_0. The set of all such metrics together with the subspace topology will be denoted by W^{loc}.

Remark 5.19 ([3, p. 17]). *The number c from the preceeding definition (5.18) can be seen as a continuous function on W^{loc}*

$$c \colon W^{loc} \to (0, T_0].$$

Proposition 5.20 ([3, p. 19]). *Let $q \geq 3$. Let $h = (1, f)$ be a metric on $D^q_{\tilde{T}}$ ($\tilde{T} > T_0$ from equation 5.14), such that there is a point $\sigma \in (0, T_0/2)$ satisfying:*

1. *$f^{(n)}(\sigma) = 0$ for all $n \geq 1$*
2. *$0 \leq f' \leq 1$ and $f'' \leq 0$ on $[0, \sigma]$.*

Then there exists a family of metrics $\Psi_2(h, \lambda, \sigma) = (f_\lambda, g_\lambda)$ on the disc $D^q_{T_0}$, such that

1. *$\Psi(h, 0, \sigma) = h$*
2. *$\Psi(h, 1, \sigma) = \left(\frac{T_0}{\sigma}\right)^2 dt^2 + \tilde{f}_0\left(\frac{T_0}{\sigma} t\right)^2 d\xi^2$ on D^q_σ*
3. *$\Psi(h, \lambda, \sigma) = h$ near the boundary of $D^q_{T(h)}$.*

Ψ depends continuously on h, λ and σ.

Remark 5.21. *Ψ_1 has been defined as the pullback under some diffeomorphisms Φ_{t_1,t_2,t_3}. Before applying them, the metric has the form $dt^2 + f^2 d\xi^2$. Therefore, the metric we start the following proof with, will be the metric $\left(1, f\left(\tilde{\psi}_{1,\alpha}(\psi_1)\right)\right)$ on $D^q_{\tilde{T}_2(1)}$ which we will abbreviate by $(1, f)$.*

Proof of Proposition 5.20. In this proof we use of the collar we created:

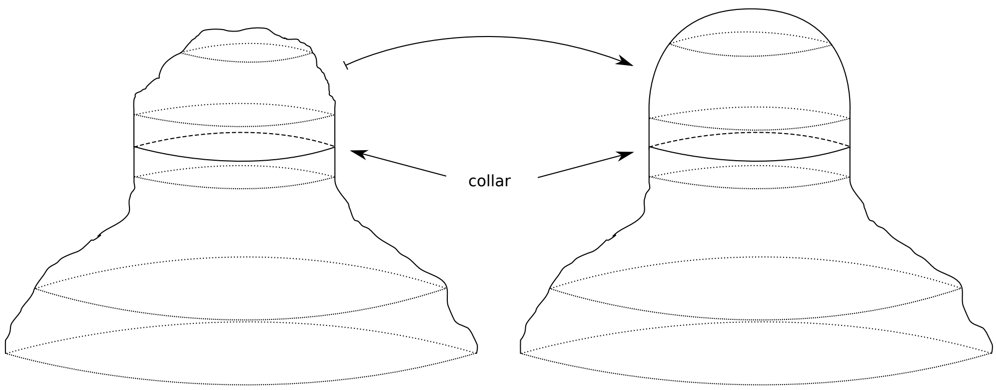


Figure 5.5: Deforming into W^{loc}

We only construct a deformation on D^q_σ. We will then prolong the collar, i.e. we will insert a cylinder of length a. This can be done because we created a collar. On the cylinder we take the straight line homotopy to the metric $h^a = g_{(t/a)} + dt^2$ from (2.68). This is just the same as in the proof of (4.13).

So let's have a look at D^q_σ. The idea is to take a straight line

$$f_\lambda = \lambda \tilde{f}_0 + (1-\lambda) f.$$

This usually doesn't preserve positive scalar curvature, but since $0 \le f' \le 1$ and $f'' \le 0$ as long as $t \in [0,\sigma]$, we have

$$\kappa = (q-1)\Big((q-2)\frac{1-\overbrace{((1-\lambda)f+\lambda\tilde{f}_0)'^2}^{\in[0,1]}}{((1-\lambda)f+\lambda\tilde{f}_0)^2} - 2\frac{\overbrace{((1-\lambda)f+\lambda\tilde{f}_0)''}^{\le 0}}{(1-\lambda)f+\lambda\tilde{f}_0}\Big) > 0.$$

Let $\nu \in \mathbb{R}$. Since the scalar curvature of the metric $\nu h = (\nu, \nu\cdot f)$ is equal to

$$\begin{aligned}
\kappa &= \frac{(q-1)}{(\nu f)^2\nu^3}\Big((q-2)(\nu^3 - (\nu f')^2\nu) - 2\nu^2 f'' f\nu\Big) \\
&= \frac{(q-1)}{\nu^2}\Big((q-2)\frac{1-f'^2}{f^2} - 2\frac{f''}{f}\Big),
\end{aligned}$$

there is a $\nu \in (0,1]$, such that $\nu \cdot f_\lambda$ has scalar curvature greater than B. So the first part of our homotopy will be $(1-\lambda+\lambda\nu)(1,f) = \big((1-\lambda+\lambda\nu),(1-\lambda+\lambda\nu)f\big)$. Next we apply the above homotopy $\nu \cdot f_\lambda$. Afterwards we need to rescale the metric:

$$\begin{aligned}
g_\lambda &= 1-\lambda+\lambda\cdot\frac{T_0}{\sigma} \\
f_\lambda &= \tilde{f}_0\big((1-\lambda+\lambda\cdot\frac{T_0}{\sigma})t\big).
\end{aligned}$$

If we abbreviate $a_\lambda := 1-\lambda+\lambda\cdot\frac{T_0}{\sigma}$, we can use formulae (5.7) and (5.15) to compute:

$$\begin{aligned}
\kappa &= \frac{(q-1)}{f_\lambda^2 g_\lambda^3}\Big((q-2)(g_\lambda^3 - f_\lambda'^2 g_\lambda) - 2f_\lambda'' f_\lambda g_\lambda\Big) \\
&= (q-1)\Big((q-2)\frac{1-\frac{f_\lambda'^2}{g_\lambda^2}}{f_\lambda^2} - 2\frac{f_\lambda''}{g_\lambda^2 f_\lambda}\Big) \\
&= (q-1)\Big((q-2)\frac{1-\frac{f_0'^2\cdot a_\lambda^2}{a_\lambda^2}}{f_0^2} - 2\frac{f_0''\cdot a_\lambda^2}{a_\lambda^2 f_0}\Big) \\
&= (q-1)\Big((q-2)\frac{1-f_0'^2}{f_0^2} - 2\frac{f_0''}{f_0}\Big) > B.
\end{aligned}$$

Putting everything together, we arrive at:

$$\Psi_2\big((1,f),\lambda\big) = \begin{cases}
(1-4\lambda+4\lambda\nu)\cdot\big(1,f\big) & \text{if } \lambda\in[0,1/4] \\
\nu\cdot\Big(1,(4\lambda-1)\tilde{f}_0+(2-4\lambda)f\big)\Big) & \text{if } \lambda\in[1/4,2/4] \\
\nu\cdot\Big((3-4\lambda)+(4\lambda-2)\cdot\frac{T_0}{\sigma}), & \\
\qquad \tilde{f}_0\big((3-4\lambda+(4\lambda-2)\cdot\frac{T_0}{\sigma})t\big)\Big) & \text{if } \lambda\in[2/4,3/4] \\
\big((4-4\lambda)\nu+(4\lambda-3)\big)\cdot\Big(\frac{T_0}{\sigma}, f\big(\frac{T_0}{\sigma}\cdot t\big)\Big) & \text{if } \lambda\in[3/4,1].
\end{cases}$$

For $\lambda = 1$ we arrive at $\Psi_2(h,1) = \left(\frac{T_0}{\sigma}\right)^2 dt^2 + \tilde{f}_0\left(\frac{T_0}{\sigma}\right)^2 d\xi^2$, i.e. $\Psi_2(h,1) \in W^{loc}$.

Outside of the cylinder we inserted, we didn't change the metric, so the metric remained unchanged near the boundary. $\square$

Corollary 5.22. *Let $q \geq 3$. There is a map*

$$\Psi\colon W_B \times I \to W_B,$$

such that

1. $\Psi(.,0) = id$
2. $\Psi(h,1) \in W_B^{loc}$
3. $\Psi(h,t) = h$ *near the boundary of* $D^q_{T_0}$.

Proof. This is accomplished by inserting the deformation above (5.20) before applying the rescaling diffeomorphisms in (5.14). $\square$

We finish this chapter with two little preparations which will be needed when we prove the main theorems.

Proposition 5.23 ([3, p. 20]). *Let $q \geq 3$ and let $D^q_{[t_1,t_2]}$ be the annulus $\{x \in D^q_{T_0}\colon \|x\| \in [t_1,t_2]\}$ and let h be a warped metric on this annulus, i.e. $h = g^2dt^2 + f^2d\xi^2$. Let $B > 0$ be a strict lower bound for the scalar curvature of h and suppose that for some $\varepsilon > 0$ we have $g = 1, f = r_0$ on $[t_1, t_1+\varepsilon] \cup [t_2-\varepsilon, t_2]$. Then there exists a continuous family of warped metrics h_λ, such that*

1. $h_0 = h$
2. $h_1 = dt^2 + r_0^2 d\xi^2$
3. *there exists a $\delta > 0$, such that $h_\lambda = h$ on $[t_1, t_1+\delta] \cup [t_2-\delta, t_2]$*
4. $\kappa_\lambda \geq B$.

Proof. We take $f_\lambda := (1-\lambda)f + \lambda r_0$. Then $(1, f_\lambda)$ maybe won't have positive scalar curvature, but since the metric (C, f_λ) has scalar curvature

$$\kappa = (q-1)\left((q-2)\frac{1-\frac{f_\lambda'^2}{C^2}}{f_\lambda^2} - 2\frac{f_\lambda''}{C^2 f_\lambda}\right),$$

there is a $C \geq 1$, such that $\kappa > 0$. The following approach is similar to the one from the proof of the preceding lemma (5.20). We first choose $\delta = \varepsilon/2$.
Since f_λ is constant on the annuli $[t_1+\delta, t_1+\varepsilon]$ and $[t_2-\varepsilon, t_2-\delta]$, we don't have to enlarge them. Then we increase $g(t)$, so that we may assume that $g \equiv \tilde{C} = \max\big(C, \max_{t\in[t_1,t_2]} g(t)\big)$ and go through the deformation f_λ defined above on $[t_1+\varepsilon, t_2-$

Figure 5.6: $[t_1, t_2]$

$\varepsilon]$. At $t \in [t_1+\delta = t_\delta, t_\varepsilon = t_1+\varepsilon]$ we take the metric $g_\lambda = (1-\lambda(\frac{t-t_\delta}{t_\varepsilon - t_\delta}))+\lambda(\frac{t-t_\delta}{t_\varepsilon - t_\delta})\tilde{C} \geq 1$. This satisfies:

$$\begin{aligned} g_\lambda(t_\delta) &= 1 \\ g_\lambda(t_\varepsilon) &= 1 - \lambda + \lambda\tilde{C} \end{aligned}$$

and has scalar curvature greater B. f_λ is constant on $[t_\delta, t_\varepsilon]$. We use the same approach on $[t_2 - \varepsilon, t_2 - \delta]$.

Last we bring $\tilde{C}$ down again. During the entire process, we didn't change the metric near the boundary and the scalar curvature was greater then 0. If we increase C even more, we can even achieve that $\kappa_\lambda > B$. It remains to verify property 2. On $[t_1+\varepsilon, t_2-\varepsilon]$ and on $[t_1, t_1+\delta] \cup [t_2-\delta, t_2]$ this is clear. On $[t_1+\delta, t_1+\varepsilon] \cup [t_2-\varepsilon, t_2-\delta]$ it follows from the fact that f_λ is constant here. □

Proposition 5.24. *Let $\Phi\colon D^q_T \to D^q_T$ be a radial diffeomorphism, i.e. $\Phi(v) = \frac{\phi(\|v\|)}{\|v\|}v$ for some diffeomorphism $\phi\colon [0,T] \to [0,T]$ with $\phi(0) = 0$. Then, the map*

$$\Phi^*\colon W_B \to W_B$$

is well defined.

Proof. It is clear that the scalar curvature of Φ^*h is the same as the scalar curvature of h. It only remains to show, Φ^*h is a warped metric. But this follows immediatly from the computation below (5.1), if we take $G^{-1}(t) = \phi$ and $\varphi = \Phi$:

Let X, Y be tangent vectors at a point $t, s \in [0,T] \times S^{q-1}$ and let $h = g^2dt^2 + f^2d\xi^2 \in W_B$. Then

$$\Phi^*h(X,Y) = h \circ (D_{(t,s)}\Phi(X), D_{(t,s)}\Phi(Y)).$$

If X is a vector in the $d\xi$-direction, $D_{(t,s)}\Phi(X) = X$, since Φ is radial. If X is in the dt-direction

$$D_{(t,s)}\Phi(X) = \frac{d}{dx}|_{x=0}\phi\big((t+x)X\big) = \phi'(t) \cdot X.$$

So we can conclude our computation with

$$\begin{aligned}
\Phi^* h(X,Y) &= \begin{cases} g(\phi(t))^2 \cdot dt^2\Big(\phi'(t)\cdot X, \phi'\cdot Y\Big) & \text{if } X,Y \text{ are in } dt\text{-direction} \\ f(\phi(t))^2 \cdot d\xi^2\big(X,Y\big) & \text{if } X,Y \text{ are in } d\xi\text{-direction} \\ 0 & \text{else} \end{cases} \\
&= \underbrace{g(\phi(t))^2 \cdot (\phi')^2}_{=:\tilde{g}^2} \cdot dt^2 + \underbrace{f(\phi(t))^2}_{=:\tilde{f}^2} d\xi^2.
\end{aligned}$$

□

6 Surgery invariance of the homotopy type of $\mathcal{R}^+(M)$

This chapter contains the results from [3, pp. 20-22].

Let $N^p \subset M^n$ be a closed submanifold with trivial normal bundle. Let g_N be a fixed metric on N and let

$$B = \max\Big(-\min_{x\in N}(\kappa_{g_N}(x)), 0\Big).$$

Let $\tau\colon N^p \times D^q_{T_0} \hookrightarrow M^n$ a fixed tubular neighbourhood of $N \subset M$. Let g_0 be a fixed torpedo metric of radius T_0, where T_0 is small enough so that $g_0 \in W_B$. Furthermore let

$$W(N,\tau) = \{g \in \mathcal{R}^+(M)\colon \tau^*(g) = g_N + g_w,\ g_w \in W_B\}.$$

The subspace $W^{loc}(N,\tau) \subset W(N,\tau)$ shall be defined analogously to (5.18).

Proposition 6.1 ([3, p. 21])**.** *Let $(g_s)_{s\in S}$ be a compact family of metrics. If $g_s \in W(N,\tau)$, then* $\mathbf{GLC}(t,s) \in W(N,\tau)$ *for all* $t \in [0,1]$.

Proof. In order to proof this, we need to take a closer look at the construction of the **GLC**-map (4.16). First we note that we can choose **GLC** to take place inside of an arbitrarily small tubular neighbourhood of N, so we may assume that the entire deformation takes place inside a small neighbourhood of $\tau(N \times D^q_{T_0})$

The first part of this deformation is to push out a cylinder:

$$f^*_{\alpha_1(\lambda),s}(g_s + dt^*).$$

Since $f_{\gamma,s}\colon M \hookrightarrow M \times \mathbb{R}$ was defined by $f_{\gamma,s} = \exp^\perp_{T_\gamma(g_s)} \circ dp^{-1}_{\gamma,g_s} \circ \phi^{-1} \circ (\exp^\perp_{g_s})^{-1}$, we know that $f_{\gamma,s}|_N$ is a diffeomorphism onto $N_\gamma = f_{\gamma,s}(N)$. Therefore, $f^*_{\gamma,s}$ doesn't change the metric on N and

$$\tau^* f^*_{\gamma,s}(g_s + dt^2) = g_N + g_{D^q_{T_0}}$$

for some metric $g_{D^q_{T_0}}$ on the disc. Now let's have a look at what happens to the metric on $D^q_{T_0}$. The map $dp^{-1}_{\gamma,g_s} \circ \phi^{-1}$ only acts by stretching of the normal bundle by some radial diffeomorphism which preserves the property of being a warped metric (5.24). Since everything that could happen while applying $(\exp^\perp_{g_s})^{-1}$ is reversed by $\exp^\perp_{T_\gamma(g_s)}$, we know that during this first deformation warped metrics remain warped.

The second part of a **GLC**-deformation consists of taking a straight path on the cylinder to the metric $g_N + g_0$ on top of the cylinder. This again means, the deformation is constant on the N-factor. So again, we have

$$\tau^* f^*_{\alpha_1(1),s}\big(\alpha_2(\lambda,s)\big) = g_N + \tilde{g}_{D^q_{T_0}}.$$

As g_0 is a warped metric, we can write $g_0 = g_1^2 dt^2 + f_1^2 d\xi^2$. By

$$\begin{aligned}
t(g_1^2dt^2 + f_1^2 d\xi^2) \quad &+ \quad (1-t)(g_2^2dt^2 + f_2^2 d\xi^2) \\
&= \quad \underbrace{(tg_1^2 + (1-t)g_2^2)}_{=\tilde{g}^2} dt^2 + \underbrace{(tf_1^2 + (1-t)f_2^2)}_{=\tilde{f}^2} d\xi^2 \\
\tilde{g} \quad &= \quad \sqrt{tg_1^2 + (1-t)g_2^2} \\
\tilde{f} \quad &= \quad \sqrt{tf_1^2 + (1-t)f_2^2}
\end{aligned}$$

we conclude that the space of warped metrics in the disc is convex. So we deduce that $\tilde{g}_{D^q_{T_0}}$ from above remains a warped metric during the second part of a **GLC**-deformation.

The third part of the deformation purely consists of taking the pullback under radial diffeomorphisms. By (5.24) it is clear that this maps $W(N,\tau)$ into itself. □

Theorem 6.2. *Let $q \geq 3$. Then the subspace $\mathcal{R}^+_0(M)$ is a weak deformation retract of $W(N,\tau)$.*

Proof. Without loss of generality we may assume that τ is defined on $N \times D^q_{T_1}$ for some $T_1 > T_0$.

We fix a continuous family of radial diffeomorphisms $(\Phi_{\lambda,T})_{\lambda\in I, T\in(0,T_0]}$ of $D^q_{T_1}$, such that:

1. $\Phi_{0,T} = id$
2. $\Phi_{1,T}$ acts on $D^q_{T_0}$ by multiplication by T/T_0
3. If $T^* \in [T_0, T_1]$ is such that $\Phi_{\lambda,T}(D^q_{T^*}) = D^q_{T_0}$, then $\Phi_{\lambda,T}$ is a radial isometry on a neighbourhood of $\partial D^q_{T^*} = S^{q-1}(T^*)$.

We define the retraction map as

$$r\colon W(N,\tau) \underset{(5.22)}{\overset{\Psi(\_,1)}{\rightarrow}} W^{loc}(N,\tau) \overset{\Phi}{\rightarrow} \mathcal{R}^+_0(M),$$

where $\Phi(h) = \Phi^*_{1,c(h)}h$, where $c(h)$ is the map from (5.19) and Ψ is the map that was constructed in chapter 5 (5.22). What we need to show is that for r and $\iota\colon \mathcal{R}^+_0(M) \hookrightarrow W(N,\tau)$, the compositions $r\circ\iota$ and $\iota\circ r$ are homotopic to the identity.

Let's consider $\iota\circ r$ first. The homotopy here is given by

$$D(h,\lambda) = \begin{cases} \Psi(h,2\lambda) & \text{if } \lambda\in[0,1/2] \\ \Phi^*_{(2\lambda-1),c(h')}h' & \text{if } \lambda\in[1/2,1],\ (h' = \Psi(h,1)). \end{cases}$$

It is clear that

$$
\begin{aligned}
D(h,0) &= h \\
D(h,1) &= \Phi^*_{1,c(\Psi(h,1))}\Psi(h,1) = r(h).
\end{aligned}
$$

Now we have to look at $r \circ \iota$. For any metric h we know from chapter (5.2) that $c(\Psi(h,1)) = T \leq T_0/2$. Let h be a metric from $\mathcal{R}_0^+(M)$. Because of our construction of Ψ (the whole construction from chapter 5.2 took place inside of $D^n_{T_0/2}$) we know that $\Psi(h,1) = h$ on $T_0/2 \leq \|x\| \leq T_1$.

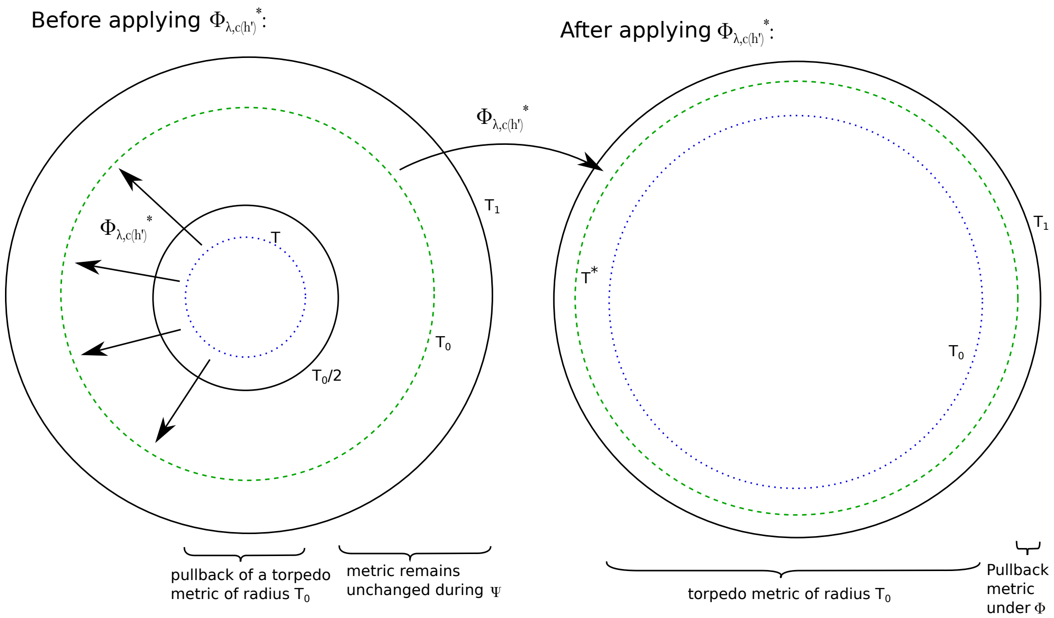


Figure 6.1: Rescaling $D^q_{T_1}$

So there is a number $T^* > T_0$ satisfying $\Phi_{1,c(h')}(D^q_{T^*}) = D^q_{T_0}$. This means that on $T^* \leq \|x\| \leq T_1$, $r(h)$ is exactly the pullback of h restricted to $T_0 \leq \|x\| \leq T_1$ under the diffeomorphism $\Phi_{1,c(h')}$. On $T_0 \leq \|x\| \leq T^*$ it is a warped metric h_w, which satisfies $h_w|_{S^{q-1}(T_0)} = h_w|_{S^{q-1}(T^*)}$, since $\Phi_{\lambda,c(h')}$ was chosen to be a radial isometry on a neighbourhood of $S^{q-1}(T^*)$. Without loss of generality we may assume that $h_w = dt^2 + r_0 d\xi^2$ on a neighbourhood of the boundary and by (5.23) we may assume that $h_w = dt^2 + r_0 d\xi^2$ on the entire annulus $T_0 \leq \|x\| \leq T^*$. The desired deformation then consists of contracting the annulus $T_0 \leq \|x\| \leq T^*$ to $S^{q-1}(T_0)$ and at the same time stretching $T^* \leq \|x\| \leq T_1$ to $T_0 \leq \|x\| \leq T_1$. This will be done by going backwards through the above family of diffeomorphisms $\Phi_{\lambda,c(h')}$, i.e. by taking the deformation $\Phi_{1-\lambda,c(h')}$. We then receive the metric h, we initially started with, which completes the proof of this theorem. □

Theorem 6.3. *Let $q \geq 3$. The inclusion map*

$$\iota\colon \mathcal{R}_0^+(M) \to \mathcal{R}^+(M)$$

is a homotopy equivalence.

Proof. We already know that $\mathcal{R}_0^+(M)$ is a deformation retract of $W(\tau, N)$. Since all of these spaces are dominated by CW-complexes [12, p. 4], we conclude from WHITEHEAD's theorem [16, p. 215] that it is sufficient to show, that the inclusion $\iota\colon W(N,\tau) \hookrightarrow \mathcal{R}^+(M)$ is a weak equivalence. So it suffices to check that ι is a bijection on π_0 and that $\pi_r\big(\mathcal{R}^+(M), W(N,\tau)\big) = 0$ for all $r \geq 1$.

We first verify that ι is a bijection on π_0. Let $h_1, h_2 \in W(N,\tau)$ be two metrics that lie in the same path component of $\mathcal{R}^+(M)$. Thus, there is a path α from h_1 to h_2. Now we need to show that there is a path in $W(N,\tau)$ from h_1 to h_2. Let

$$\mathbf{GLC}\colon D^1 \times I \to \mathcal{R}^+(M)$$

be a GROMOV-LAWSON-CHERNYSH-deformation for the family $\alpha(s)$ of metrics, i.e a homotopy satisfying $\mathbf{GLC}(s,0) = \alpha(s)$ and $\mathbf{GLC}(s,1) \in \mathcal{R}_0^+(M) \subset W(N,\tau)$ for all $s \in D^1$. Then

$$\tilde{\alpha}(t) = \begin{cases} \mathbf{GLC}(0, 3t) & \text{if } t \in [0, 1/3] \\ \mathbf{GLC}(3t-1, 1) & \text{if } t \in [1/3, 2/3] \\ \mathbf{GLC}(1, 3-3t) & \text{if } t \in [2/3, 1] \end{cases}$$

is our required path.

Now let $[\alpha] \in \pi_r\big(\mathcal{R}^+(M), W(N,\tau)\big)$, where $\alpha\colon (D^r, S^{r-1}) \to \big(\mathcal{R}^+(M), W(N,\tau)\big)$ is a continuous map. Let $\mathbf{GLC}$ be a deformation for the family $\alpha(s)$ of metrics. Then

$$\mathbf{GLC}\colon D^r \times I \to \mathcal{R}^+(M)$$

is a deformation, such that $\mathbf{GLC}(s,1) \in \mathcal{R}_0^+(M) \subset W(N,\tau)$ and $\mathbf{GLC}(S^{r-1}, t) \in W(N,\tau)$ for all $t \in I$. This implies $[\alpha] = 0$. $\square$

Theorem 6.4. *Let M_1 and M_2 be two manifolds, such that M_2 is obtained from M_1 by surgery in dimension of at least 2 and in codimension of at least 3. Then there is a homotopy equivalence between $\mathcal{R}^+(M)$ and $\mathcal{R}^+(M_1)$.*

Proof. Let $p \geq 2$, $q \geq 3$ and let $\tau\colon S^p \times D^q \hookrightarrow M_1^n$ be the embedding of the surgery sphere with trivial normal bundle. Furthermore let

$$\begin{aligned} M_0 &:= M_1 \backslash \tau(S^p \times D^q) \\ M_1 &= M_0 \underset{S^p \times S^{q-1}}{\cup} (S^p \times D^q) \\ M_2 &= M_0 \underset{S^k \times S^{q-1}}{\cup} (D^{p+1} \times S^{q-1}). \end{aligned}$$

We now choose torpedometrics on D^q and on D^{p+1} of the same radius. In this case the metrics on $S^p \times D^q$ and on $D^{p+1} \times S^{q-1}$ agree on $S^p \times S^{q-1}$. Their restriction shall be denoted by h. Finally we get

$$\begin{aligned} \mathcal{R}^+(M_1) &\simeq \mathcal{R}_0^+(M_1) \\ &\simeq \{g \in \mathcal{R}^+(M_0)\colon g = h \text{ on } S^p \times S^{q-1}\} \\ &\simeq \mathcal{R}_0^+(M_2) \simeq \mathcal{R}^+(M_2). \end{aligned}$$

Note that we need $q \geq 3$ for the first equivalence and $p + 1 \geq 3$ for the last one, i.e. the surgery has to take place in dimension of at least 2 and in codimension of at least 3. This completes the proof of the main theorem. $\square$

List of Figures

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